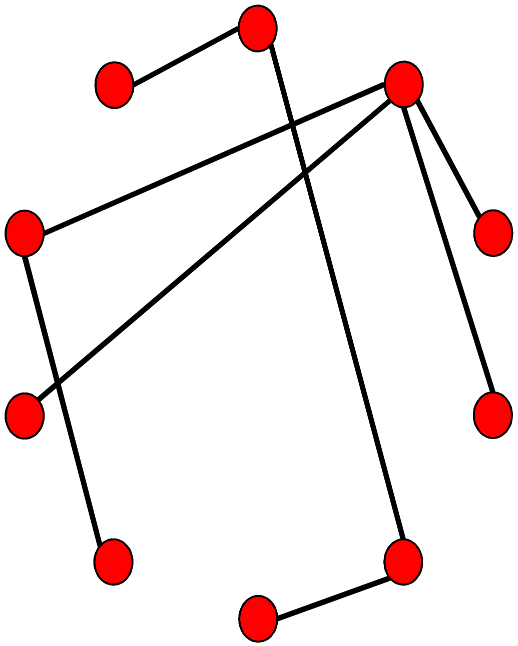
Real networks vs random networks

RANDOM NETWORK MODEL

Pául Erdös
(1913-1996)

Erdös-Rényi model (1960)

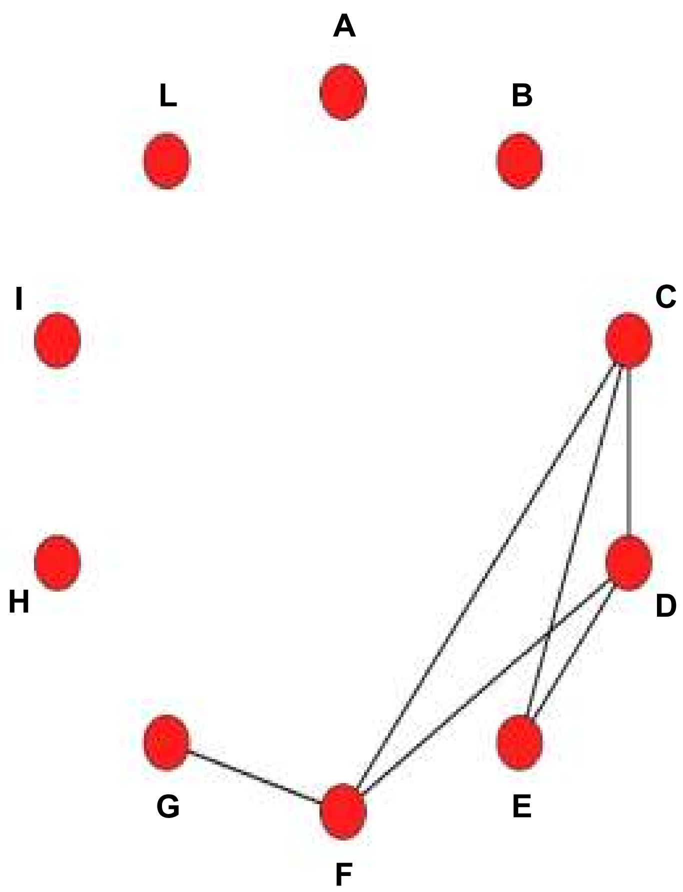


Connect with probability p

p=1/6 N=10

⟨k⟩ ~ 1.5

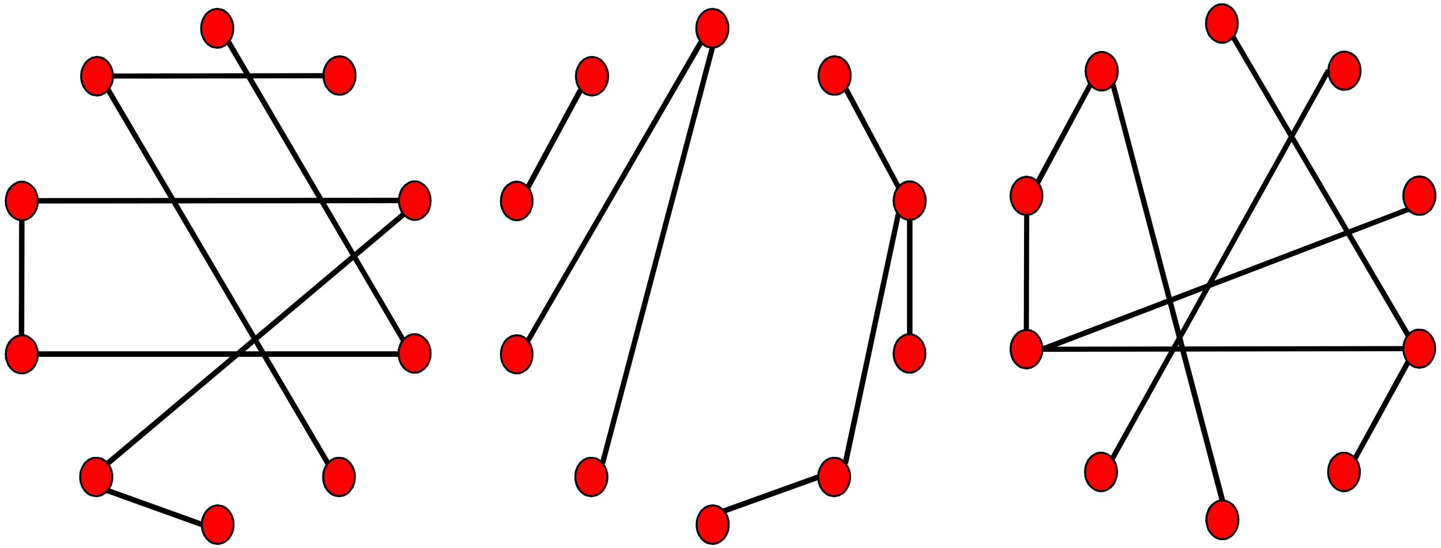
RANDOM NETWORK MODEL



Definition: A **random graph** is a graph of N labeled nodes where each pair of nodes is connected by a preset probability **p**.

RANDOM NETWORK MODEL

N and ***p*** do not uniquely define the network– we can have many different realizations of it. **How many?**



N=10
p=1/6

The probability to form a *particular* graph **G(N,L)** is

$$P(G(N,L)) = p^L (1-p)^{\frac{N(N-1)}{2} - L}$$

That is, each graph **G(N,L)** appears with probability **P(G(N,L))**.

RANDOM NETWORK MODEL

P(L): the probability to have a network of exactly *L* links

$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2} - L}$$

•The average number of links <*L*> in a random graph

$$<L> = \sum_{L=0}^{\frac{N(N-1)}{2}} L P(L) = p \frac{N(N-1)}{2}$$

$$<k> = 2L/N = p(N-1)$$

•The standard deviation

$$\sigma^2 = p(1-p)\frac{N(N-1)}{2}$$

RANDOM NETWORK MODEL

P(L): the probability to have exactly *L* links in a network of *N* nodes and probability *p*:

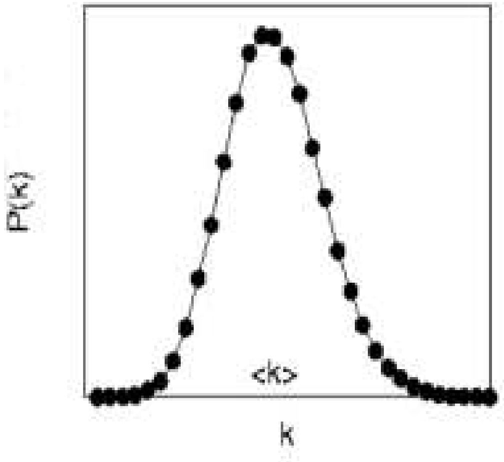
The maximum number of links in a network of N nodes.

$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2} - L}$$

Number of different ways we can choose L links among all potential links.

Binomial distributio

DEGREE DISTRIBUTION OF A RANDOM GRAPH



$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

Select k nodes from N-1

probability of having *k* edges

probability of missing N-1-k edges

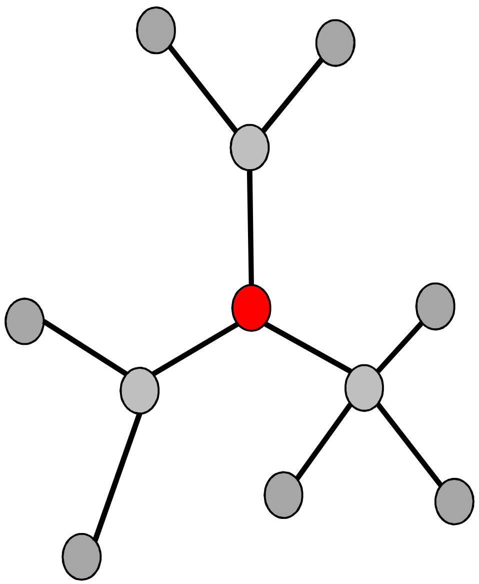
$$<k>=p(N-1) \qquad \sigma_k^2 = p(1-p)(N-1)$$

$$\frac{\sigma_k}{<k>} = \left[\frac{1-p}{p}\frac{1}{(N-1)}\right]^{1/2} \approx \frac{1}{(N-1)^{1/2}}$$

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of <k>.

DISTANCES IN RANDOM GRAPHS

Random graphs tend to have a tree-like topology with almost constant node degrees.



- nr. of first neighbors: $N_1 \cong \langle k \rangle$
- nr. of second neighbors: $N_2 \cong \langle k \rangle^2$
- nr. of neighbours at distance d: $N_d \cong \langle k \rangle^d$
- estimate maximum distance:

$$1 + \sum_{l=1}^{l_{max}} \langle k \rangle^i = N \quad \Rightarrow \quad l_{max} = \frac{\log N}{\log \langle k \rangle}$$

DISTANCES IN RANDOM GRAPHS

compare with real data

$$l_{max} = \frac{\log N}{\log\langle k\rangle}$$

Network	Size	$\langle k\rangle$	l	l_{rand}	C	C_{rand}	Reference	Nr
www, site level, undir	153127	35.21	3.1	3.35	0.1078	0.00023	Adamic, 1999	1
Internet, domain level	3015-6209	3.52-4.11	3.7-3.76	6.36-6.18	0.18-0.3	0.001	Yook e al., 2001a, Pastor-Satorras et al., 2001	2
Movie actors	225226	61	3.65	2.99	0.79	0.00027	Watts and Strogatz,1998	3
LANL co-authorship	52909	9.7	5.9	4.79	0.43	1.8×10^{-4}	Newman, 2001a, 2001b, 2001c	4
MEDLINE co-authorship	1520251	18.1	4.6	4.91	0.066	1.1×10^{-5}	Newman, 2001a, 2001b, 2001c	5
SPIRES co-authorship	56627	173	4.0	2.12	0.726	0.003	Newman, 2001a, 2001b, 2001c	6
NCSTRL co-authorship	11994	3.59	9.7	7.34	0.496	3×10^{-4}	Newman, 2001a, 2001b, 2001c	7
Math. co-authorship	70975	3.9	9.5	8.2	0.59	5.4×10^{-5}	Barabasi et al, 2001	8
Neurosci. co-authorship	209293	11.5	6	5.01	0.76	5.5×10^{-5}	Barabasi et al, 2001	9
E. coli, sustrate graph	282	7.35	2.9	3.04	0.32	0.026	Wagner and Fell, 2000	10
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09	Wagner and Fell, 2000	11
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06	Montoya and Sole, 2000	12
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03	Montoya and Sole, 2000	13
Words, co-occurrence	460902	70.13	2.67	3.03	0.437	0.0001	Ferrer i Cancho and Sole, 2001	14
Words, synonyms	22311	13.48	4.5	3.84	0.7	0.0006	Yook et al. 2001b	15
Power grid	4941	2.67	18.7	12.4	0.08	0.005	Watts and Strogatz, 1998	16
C.Elegans	282	14	2.65	2.25	0.28	0.05	Watts and Strogatz, 1998	17

Given the huge differences in scope, size, and average degree, the agreement is excellent.

Erdös-Rényi MODEL (1960)

•**Degree distribution**

Binomial, Poisson (exponential tails)

•**Clustering coefficient**

Vanishing for large network sizes

•**Average distance among nodes**

Logarithmically small

Are real networks like random graphs?

ARE REAL NETWORKS LIKE RANDOM GRAPHS?

As quantitative data about real networks became available, we can compare their topology with the predictions of random graph theory.

Note that once we have N and <k> for a random network, from it we can derive every measurable property. Indeed, we have:

Average path length:

$$< l_{rand} > \approx \frac{\log N}{\log \langle k \rangle}$$

Clustering Coefficient:

$$C_{rand} = p = \frac{\langle k \rangle}{N}$$

Degree Distribution:

$$P_{rand}(k) \cong C_{N-1}^{k} p^{k} (1-p)^{N-1-k}$$

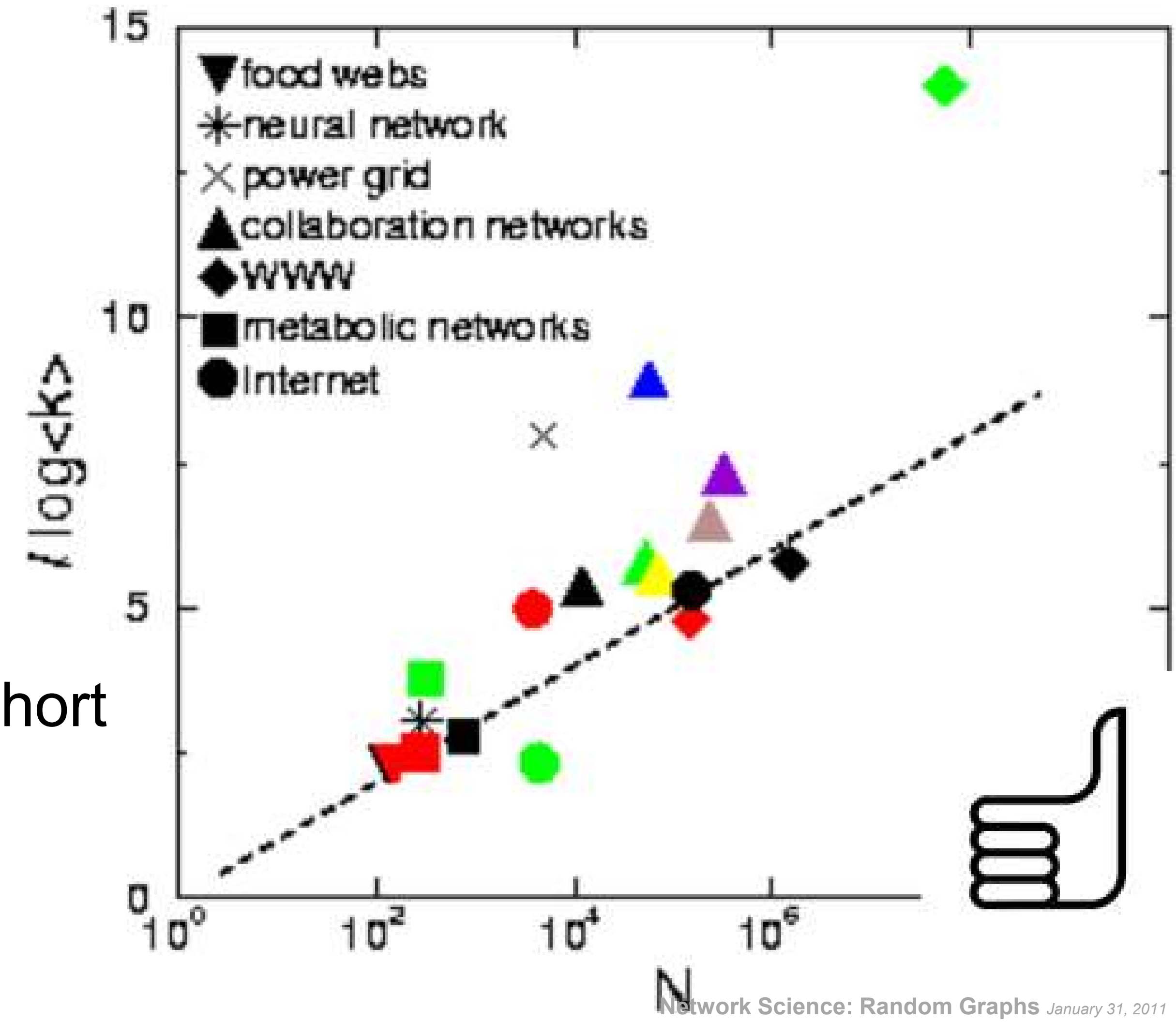
$$P(k) = e^{-<k>} \frac{<k>^{k}}{k!}$$

PATH LENGTHS IN REAL NETWORKS

Prediction:

$$l_{rand} = \frac{\log N}{\log\langle k \rangle}$$

Data:



Real networks have short distances
like random graphs.

CLUSTERING COEFFICIENT

Prediction:

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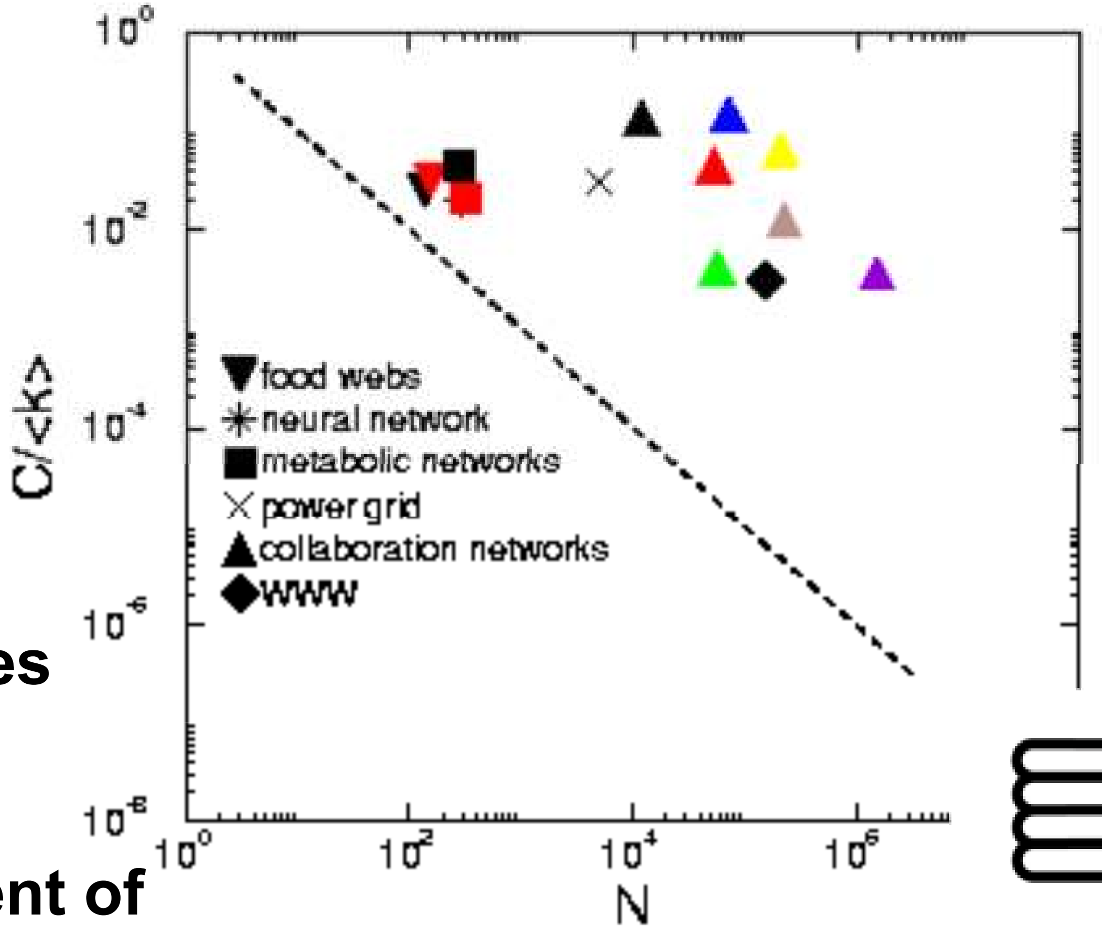
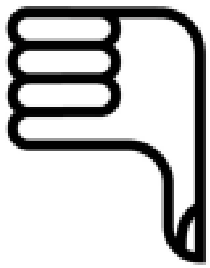




C_{rand} underestimates with orders of magnitudes the clustering coefficient of real networks.

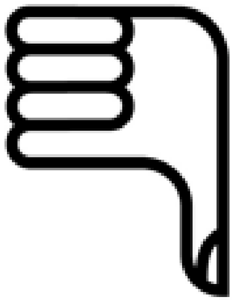
THE DEGREE DISTRIBUTION

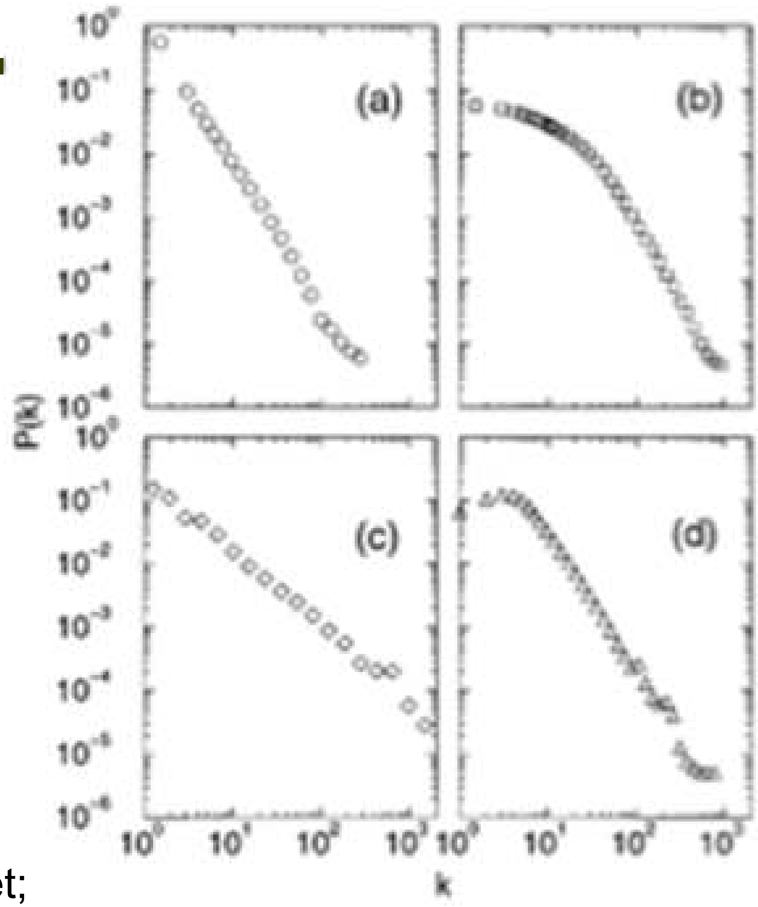
Prediction:

$$P_{rand}(k) \cong C^{k}_{N-1} p^{k} (1-p)^{N-1-k}$$

Data:

$$P(k) \approx k^{-\gamma}$$





(a) Internet;
(b) Movie Actors;
(c) Coauthorship, high energy physics;
(d) Coauthorship, neuroscience

ARE REAL NETWORKS LIKE RANDOM GRAPHS?

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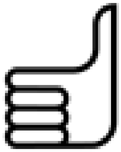
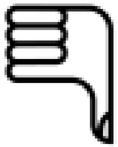


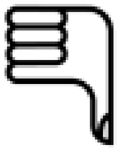
Clustering Coefficient:

$$C_{rand} = p = \frac{\langle k \rangle}{N}$$



Degree Distribution:

$$P_{rand}(k) \cong C_{N-1}^{k} p^{k} (1-p)^{N-1-k}$$

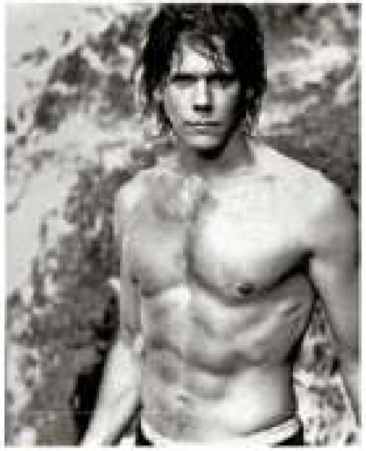


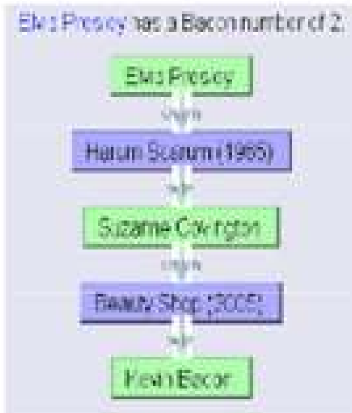
Social network as Small World

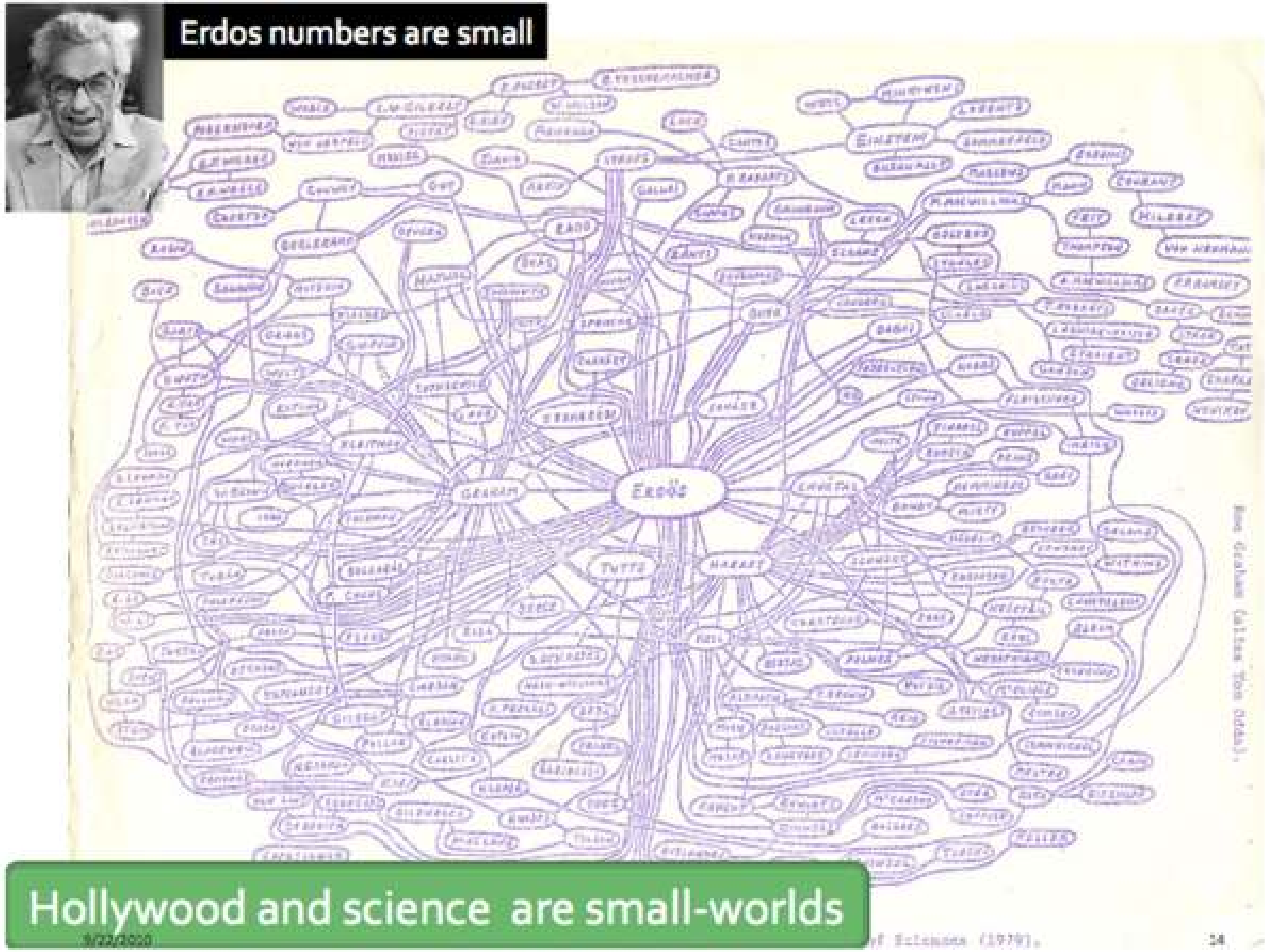
Six Degrees of Kevin Bacon

Origins of a small-world idea:

- Bacon number:
  - Create a network of Hollywood actors
  - Connect two actors if they co-appeared in the movie
  - Bacon number: number of steps to Kevin Bacon
- As of Dec 2007, the highest (finite) Bacon number reported is 8
- Only approx. 12% of all actors cannot be linked to Bacon






Erdos numbers are small
Hollywood and science are small-worlds
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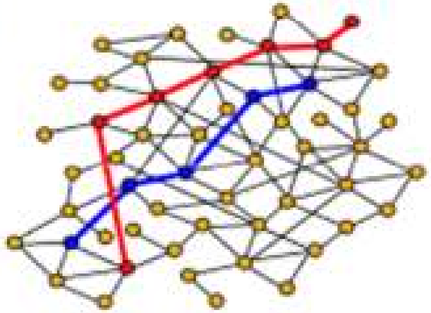
14

The Small-world experiment

Stanley Milgram

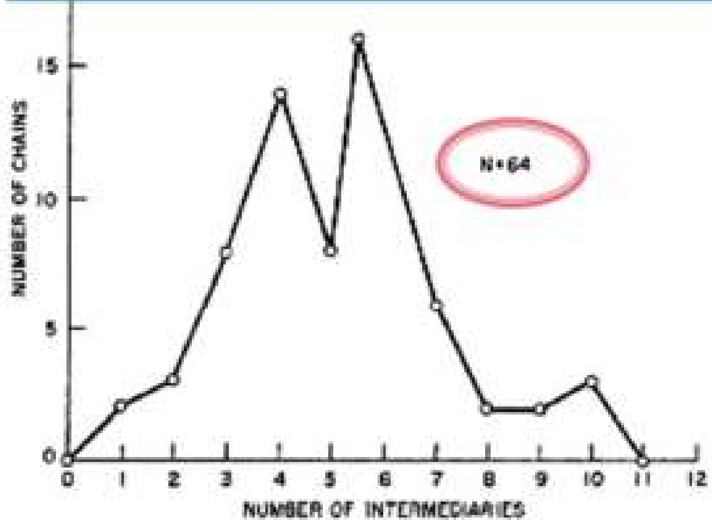


- What is the typical shortest path length between any two people?
  - Experiment on the global friendship network
    - Can't measure, need to probe explicitly
- The Small-world experiment [Stanley Milgram '67]
  - Picked 300 people at random
  - Ask them to get a letter to a by passing it through friends to a stockbroker in Boston
- How many steps does it take?

The Small-world experiment

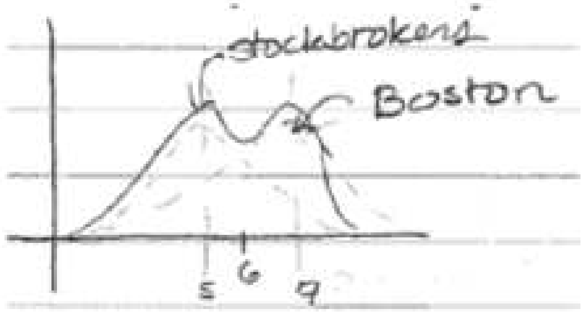
- 64 chains completed:
  - 6.2 on the average, thus "6 degrees of separation"
- Further observations:
  - People what owned stock had shortest paths to the stockbroker than random people: 5.4 vs. 5.7
  - People from the Boston area have even closer paths: 4.4

Milgram's small world experiment



Milgram: Further observations

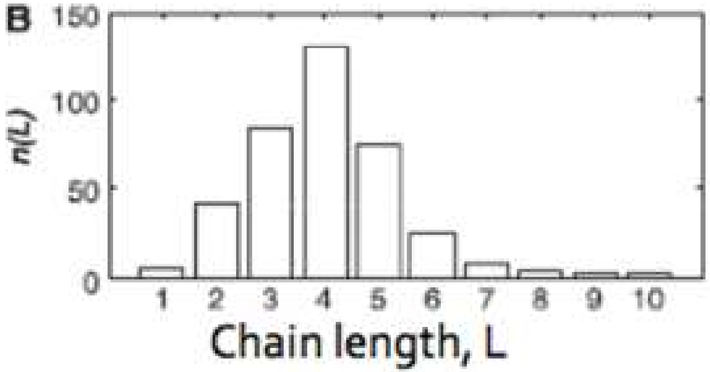
- People use different networks: Boston vs. occupation
- Criticism:
  - Funneling:
    - 31 of 64 chains passed through 1 of 3 people ass their final step → Not all links/nodes are equal
  - Choice of starting points and the target were non-random
  - People refuse to participate (25% for Milgram)
  - Some sort of social search: People in the experiment follow some strategy (e.g., geographic routing) instead of forwarding the letter to everyone. They are not finding the shortest path.
  - There are not many samples.
  - People might have used extra information resources.



[Dodds-Muhamad-Watts, '03]

Columbia small-world study

- In 2003 Dodds, Muhamad and Watts performed the experiment using email:
  - 18 targets of various backgrounds
  - 24,000 first steps (~1,500 per target)
  - 65% dropout per step
  - 384 chains completed (1.5%)



Avg. chain length = 4.01

PROBLEM: Huge drop-out rate, i.e., longer chains are less likely to complete

Correcting for the drop-out rate

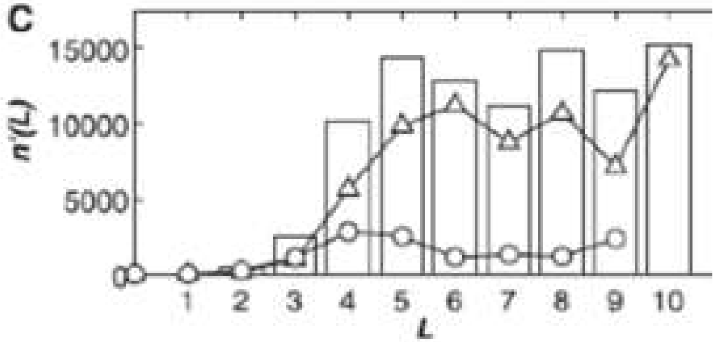
- Huge drop-out rate:
  - Longer chains don't complete

Correction proposed by Harrison-White. Let:

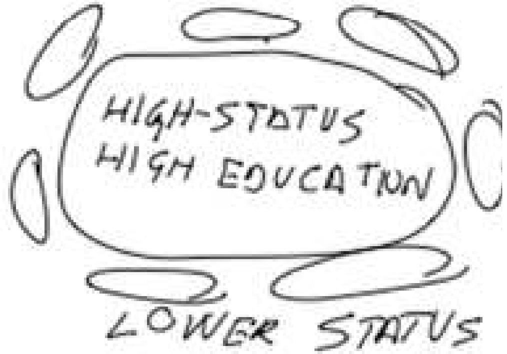
- f_j = true (unobserved) fraction of chains that would have length j
- N = total # of starters
- N_j = # starters who reached target in j steps
- Then: $f_j^* := N_j/N$
- Assume drop-out rate 1-α in each step, so $f_j^* := f_j\,\alpha^j$
- $\sum_j f_j = 1 \rightarrow \sum_j f_j^*\,\alpha^{j} = 1$
- Observe f_j^*, calculate the average dropout rate 1-α and

then $f_j = f_j^* \cdot \alpha^{-j}$

Small-world in soc. networks



- After the correction:
  - Typical path length L=7 (MEDIAN)
- Some not well understood phenomena in social networks:
  - Funneling effect: some target's friends are more likely to be the final step.
    - Conjecture: High reputation/authority
  - Effects of target's characteristics: structurally why are high-status target easier to find
    - Conjecture: Core-periphery net structure



18 target persons: Status/Authority

Target	*City*	*Country*	*Occupation*	*Gender*	*N*	N_c (%)	*r* (%)	<L>
1	Novosibirsk	Russia	PhD student	F	8234	20(0.24)	64 (76)	4.05
2	New York	USA	Writer	F	6044	31 (0.51)	65 (73)	3.61
3	Bandung	Indonesia	Unemployed	M	8151	0	66 (76)	n/a
4	New York	USA	Journalist	F	5690	41 (0.77)	60 (72)	3.9
5	Ithaca	USA	Professor	M	5655	168 (2.87)	54 (71)	3.84
6	Melbourne	Australia	Travel Consultant	F	5507	20 (0.36)	60 (71)	5.2
7	Bardufoss	Norway	Army veterinarian	M	4343	16 (0.37)	63 (76)	4.25
8	Perth	Australia	Police Officer	M	4485	4 (0.09)	64 (75)	4.5
9	Omaha	USA	Life Insurance Agent	F	4562	2 (0.04)	66 (79)	4.5
10	Welwyn Garden City	UK	Retired	M	6593	1 (0.02)	68 (74)	4
11	Paris	France	Librarian	F	4198	3 (0.07)	65 (75)	5
12	Tallinn	Estonia	Archival inspector	M	4530	8 (0.18)	63(79)	4
13	Munich	Germany	Journalist	M	4350	32 (0.74)	62 (74)	4.66
14	Split	Croatia	Student	M	6629	0	63 (77)	n/a
15	Gurgaon	India	Technology Consultant	M	4510	12 (0.27)	67 (78)	3.67
16	Managua	Nicaragua	Computer analyst	M	6547	2 (0.03)	68 (78)	5.5
17	Katikati	New Zealand	Potter	M	4091	12 (0.3)	62 (74)	4.33
18	Elderton	USA	Lutheran Pastor	M	4438	9 (0.21)	68 (76)	4.33
Totals					98,847	384 (0.4)	63 (75)	4.05

HIGH STATUS

- N… # people assigned to correspond to target
- N_c…# completed chains
- r… frac. of people who did not forward
- L… mean path length

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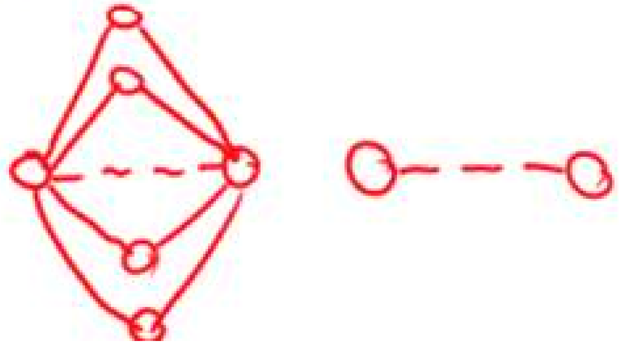
6-degrees: Should we be surprised?

- Assume each human is connected to 100 other people:
- So:
  - In step 1 she can reach 100 people
  - In step 2 she can reach 100*100 = 10,000 people
  - In step 3 she can reach 100*100*100 = 100,000 people
  - In 5 steps she can reach 10 billion people
- What's wrong here?
  - Many edges are local ("short"): friend of a friend



Planetary-Scale Views on an Instant-Messaging Network

*Jure Leskovec†

Machine Learning DepartmentCarnegie Mellon University Pittsburgh, PA, USAEric HorvitzMicrosoft Research Redmond, WA, USAMicrosoft Research Technical Report MSR-TR-2006-186June 2007

IM experiment

- Contact (buddy) list
- Messaging window

Data statistics

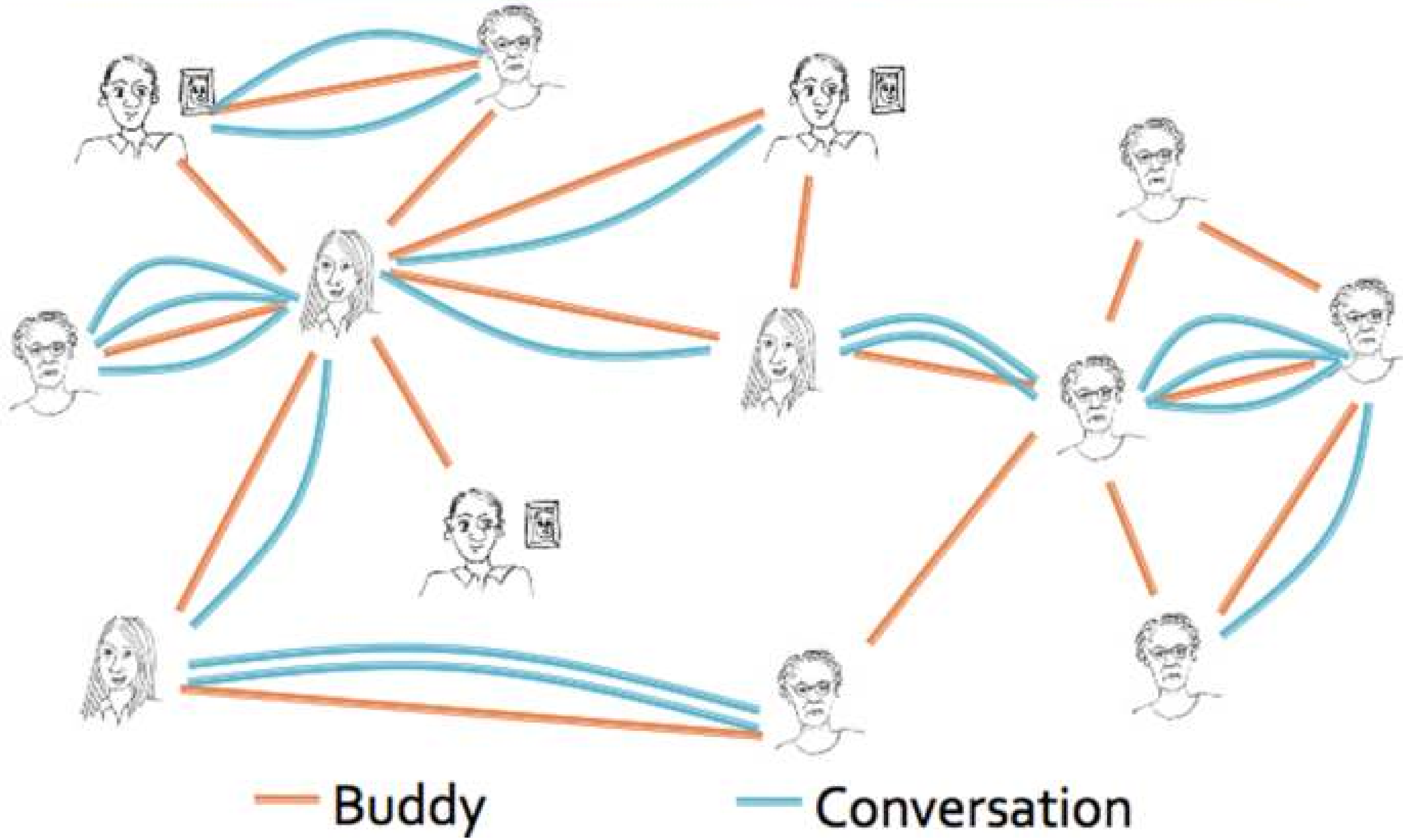
- Data for **June 2006**
- Log size:
  150Gb/day (compressed)
- Total: 1 month of communication data:
  4.5Tb of compressed data
- **Activity over June 2006 (30 days)**
  - 245 million users logged in
  - 180 million users engaged in conversations
  - 17,5 million new accounts activated
  - More than 30 billion conversations
  - More than 255 billion exchanged messages

Data statistics: typical day

Activity on a typical day (June 1 2006):

- 1 billion conversations
- 93 million users login
- 65 million different users talk (exchange messages)
- 1.5 million invitations for new accounts sent

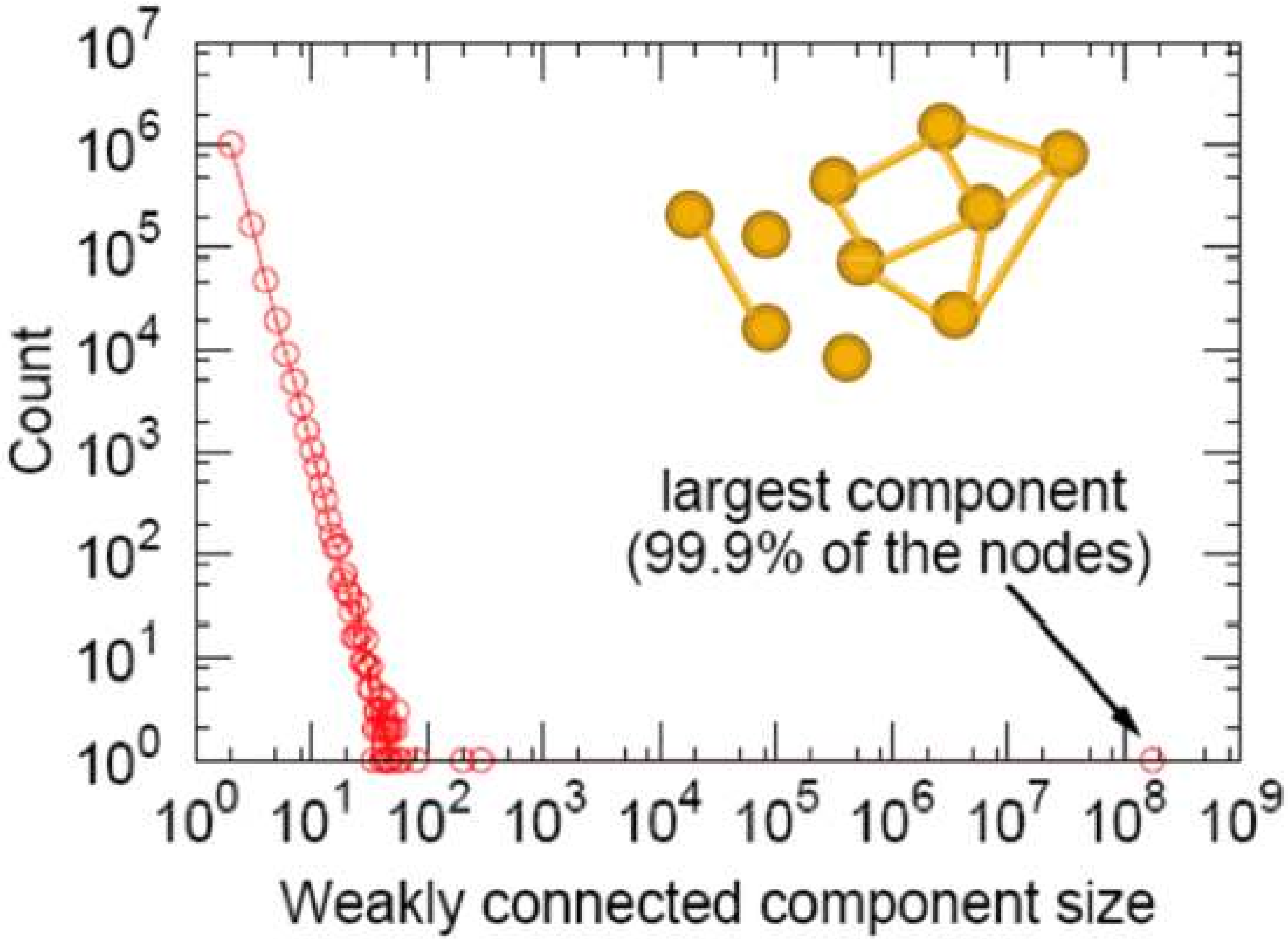
Messaging as a network



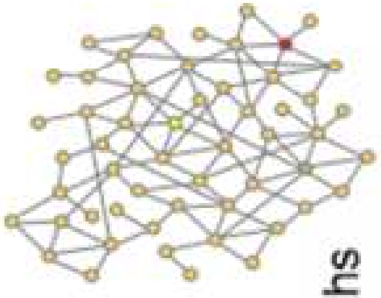
IM communication network

- Buddy graph
  - 240 million people (people that login in June '06)
  - 9.1 billion buddy edges (friendship links)
- Communication graph (take only 2-user conversations)
  - Edge if the users exchanged at least 1 message
  - 180 million people
  - 1.3 billion edges
  - 30 billion conversations

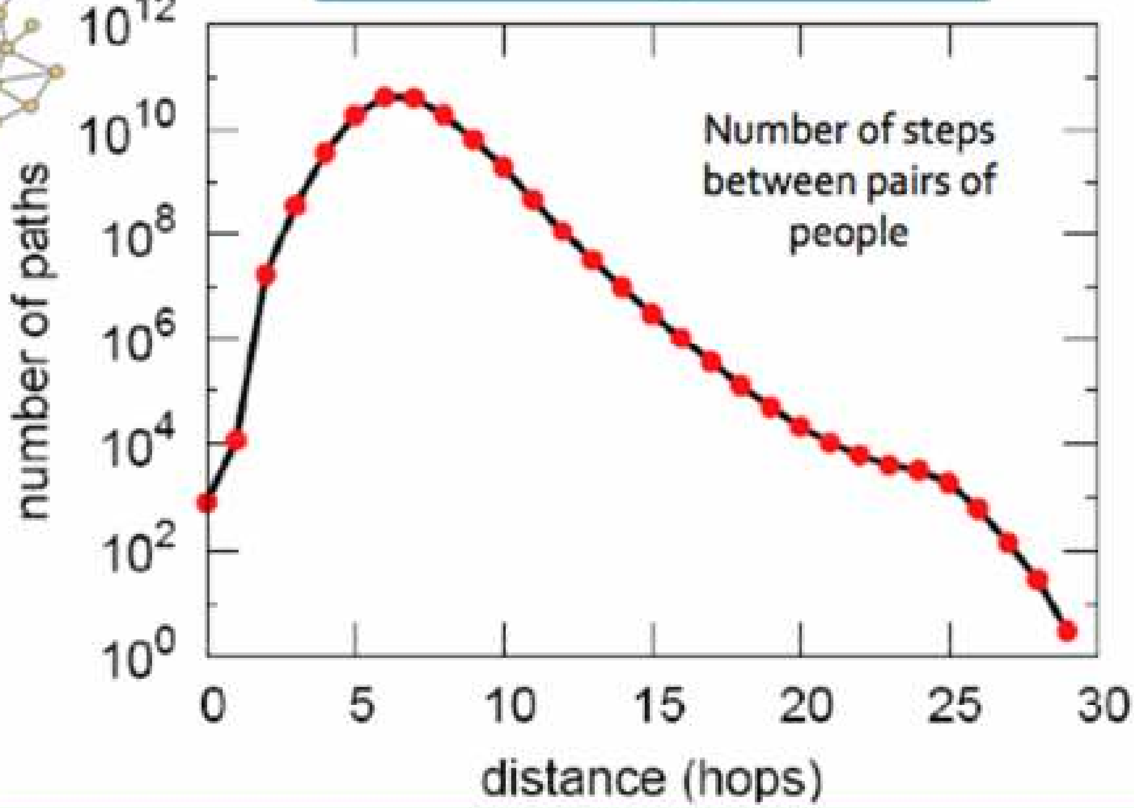
Network connectivity



MSN Network: Small world



MSN Messenger network



Hops	Nodes
0	1
1	10
2	78
3	3,96
4	8,648
5	3,299,252
6	28,395,849
7	79,059,497
8	52,995,778
9	10,321,008
10	1,955,007
11	518,410
12	149,945
13	44,616
14	13,740
15	4,476
16	1,542
17	536
18	167
19	71
20	29
21	16
22	10
23	3
24	2
25	3

Avg. path length **6.6**
90% of the people can be reached in < 8 hops