More on Lagrangian relaxation

(Wolsey : Chapter 10)

Consider an integer optimization problem of form :

$$(P) \qquad z = \max cx$$
$$Dx \leq d \quad \text{"complicating constraints"}$$
$$x \in X \quad \leftarrow \text{include the integrality constraints}$$

Let D $m \times n$ matrix
$d \in \mathbb{R}^m$
$x \in \mathbb{R}^n$

For any $u = (u_1, \ldots u_m) \geq 0$, consider the Lagrangian relaxation:

$$(P_u) \qquad z(u) = \max cx + u(d - Dx)$$
$$x \in X$$

u : <u>price</u> or <u>dual variables</u> or <u>Lagrangian multipliers</u> associated with $Dx \leq d$

<u>Proposition</u> : (P_u) is a relaxation of (P) for all $u \geq 0$.

<u>Proof</u>:

(i) $\{x : Dx \leq d, x \in X\} \subseteq X$

the feasible region of (P_u) is at least as large

(ii) $$cx + \underbrace{u}_{\geq 0}\underbrace{(d - Dx)}_{\geq 0} \geq cx \quad \forall x \in X$$

$\forall x \in X$ the objective value in (P_u) is at least as great as in (P)

$\square$

<u>Therefore</u> : $\boxed{z(u) \geq z \qquad \forall u \geq 0}$

(maximization problem)

To find the best ($\equiv$ smallest) upper bound $z(u)$ for $u \geq 0$, we solve the Lagrangian Dual Problem:

$$(LD) \qquad w_{LD} = \min \{ z(u) : u \geq 0 \}$$

Solving a Lagrangian relaxation may sometimes lead to solve (P):

Let $u \geq 0$:

Proposition: If:

(i) $x(u)$ is an optimal solution to (P_u)

(ii) $D\,x(u) \leq d$

(iii) if $u_i > 0$ then $(D\,x(u))_i = d_i$

(complementarity)

then $x(u)$ is optimal for (P).

Proof:

From (i):

$$w_{LD} \leq z(u) = c\,x(u) + u\,(d - D\,x(u))$$

From (iii):

$$c\,x(u) + \underbrace{u(d - D\,x(u))}_{=0} = c\,x(u)$$

From (ii):

$c\,x(u) \leq z$ since $x(u)$ is feasible for (P)

It follows that:

$$w_{LD} \leq c\,x(u) \leq z$$

Since $w_{LD} \geq z$ (it provides an upper bound)

we get $w_{LD} = c\,x(u) = z$.

So, $x(u)$ is optimal for (P) □

<u>Obs 1:</u> in this case u is optimal for (LD)

<u>Obs 2:</u> if the constraints "dualized" (i.e. $Dx \leq d$) are equality constraints (i.e. $Dx = d$) then (iii) is always satisfied.

In this case, an optimal solution $x(u)$ to (P_u) is optimal for (P) if it is feasible for (P).

<u>example</u> (<u>UFL</u>)

consider the strong formulation (UFL_2)

max in Walsey

$$z = \min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

<u>dualize</u> $\longrightarrow$ $\sum_{j \in N} x_{ij} = 1 \quad \forall i \in M \qquad (u_i \gtrless 0)$

$$(P) \qquad \begin{aligned} & x_{ij} - y_j \le 0 && \forall i \in M, j \in N \\ & x_{ij} \ge 0 && \forall i \in M, j \in N \\ & y_j \in \{0,1\} && \forall j \in N \end{aligned}$$

$$(P_u) \qquad z(u) = \min \sum_{i \in M} \sum_{j \in N} (c_{ij} - u_i) x_{ij} + \sum_{j \in N} f_j \cdot y_j + \sum_{i \in M} u_i$$

$$\begin{aligned} & x_{ij} \le y_j && \forall i \in M, j \in N \\ & x_{ij} \ge 0 && \forall i \in M, j \in N \\ & y_j \in \{0,1\} && \forall j \in N \end{aligned}$$

(P_u) decomposes into n subproblems, one for location:

$$z(u) = \sum_{j \in N} z_j(u) + \sum_{i \in M} u_i$$

where

$$(P_u^j) \quad z_j(u) = \min \sum_{i \in M} (c_{ij} - u_i) x_{ij} + f_j \cdot y_j$$

$$x_{ij} \leq y_j \quad \forall i \in M$$

$$x_{ij} \geq 0 \quad \forall i \in M$$

$$y_j \in \{0, 1\}$$

(P_u^j) can be solved <u>by inspection</u>:

1) if $y_j = 0$ then $x_{ij} = 0 \quad \forall i \in M$, so

$$z_j^1(u) = 0$$

2) if $y_j = 1$:

$$x_{ij} = \begin{cases} 1 & \text{if } c_{ij} - u_i < 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in M$$

So

$$z_j^2(u) = \sum_{i \in M} \min\{c_{ij} - u_i, 0\} + f_j$$

Therefore :

$$z_j(u) = \min\Big\{ \underset{y_j = 0}{\overset{z_j^1(u)}{0}} \, , \; \underset{y_j = 1}{\overset{z_j^2(u)}{\sum_{i \in M} \min\{c_{ij} - u_i , 0\} + f_j}} \Big\}$$

$\forall j \in N$

< easily solvable ! >

<u>example</u> $m = 6$ $n = 5$

$$f = (2, 4, 5, 3, 3)$$

$$(c_{ij}) = \begin{bmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{bmatrix}$$

If we choose :

$$u = (5 \;\; 6 \;\; 3 \;\; 2 \;\; 5 \;\; 4)$$

Modified costs:

$$(c_{ij}-u_i) = \begin{array}{ccccc} & \text{location } 2 & & & \\ & \Downarrow & & & \end{array}$$

$$\begin{bmatrix} 1 & -3 & -4 & -2 & 0 \\ -2 & 4 & -4 & 0 & -5 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -1 & 2 \\ -4 & 3 & 1 & -3 & 0 \\ -1 & -2 & 0 & 4 & -3 \end{bmatrix}$$

$$\sum_{i \in M} u_i = 25$$

for $j = 2$

- if $y_2 = 0$ then $z_2^1(u) = 0$
- if $y_2 = 1$ then $x_{12} = 1$, $x_{32} = 1$, $x_{42} = 1$, $x_{62} = 1$

$$z_2^2(u) = -3 - 1 - 2 - 2 + \underset{\underset{4}{\shortparallel}}{f_2} = -4$$

So it is optimal to set

$y_2 = 1$ giving $z_2(u) = -4$

We have to perform a similar calculation for each location to find $z(u)$:

$$z(u) = z_1(u) + \underbrace{z_2(u)}_{= -4} + z_3(u) + z_4(u) + z_5(u) + \underbrace{\sum_{i \in N} u_i}_{25}$$

For other u?

- How can we find the best Lagrangian bound?

- How good is such an upper bound, i.e. ω_{LD}?

Strength of the Lagrangian Dual

Let (P) $\quad z = \max cx$

$$Dx \leq d$$

$$x \in X$$

$$(P_u) \quad z(u) = \max cx + u(d - Dx)$$

$$x \in X$$

$u \geq 0$

$$(LD) \quad w_{LD} = \min \{ z(u) : u \geq 0 \}$$

Suppose for simplicity that $X = \{x_1, \dots x_T\}$, i.e. finite set of points:

$$w_{LD} = \min_{u \geq 0} \left\{ \max_{x \in X} cx + u(d - Dx) \right\}$$

$$= \min_{u \geq 0} \left\{ \max_{t = 1, .. T} cx_t + u(d - Dx_t) \right\}$$

$$= \min_{u \geq 0} \left\{ \underbrace{\max_{t = 1, .. T} cx_t + u(d - Dx_t)}_{\text{maximum of } T \text{ values}} \right\}$$

Let us introduce an auxiliary variable η to estimate the maximum

$$w_{LD} = \min \eta$$

$$(\lambda_t) \qquad \eta \geq \overset{\text{constant}}{c x_t} + u(d - D x_t) \quad t = 1, \dots T$$

$$u \geq 0$$

The latter is a Linear Programming problem. By the previous assumption on X it has finite optimum value. Therefore, from strong duality we can replace it by its dual:

$$w_{LD} = \max \sum_{t=1}^{T} \lambda_t (c x_t)$$

$$\sum_{t=1}^{T} \lambda_t (D x_t - d) \leq 0$$

$$\sum_{t=1}^{T} \lambda_t = 1$$

$$\lambda_t \geq 0 \qquad t = 1, \dots T$$

Setting $x = \sum_{t=1}^{T} \lambda_t x_t$ s.t. $\sum_{t=1}^{T} \lambda_t = 1$, $\lambda_t \geq 0$ $t = 1, \dots T$

we get

$$w_{LD} = \max c \cdot \underbrace{\sum_{t=1}^{T} \lambda_t x_t}_{x}$$

$$D \cdot \underbrace{\sum_{t=1}^{T} \lambda_t x_t}_{x} - d \cdot \underbrace{\sum_{t=1}^{T} \lambda_t}_{\overset{\shortparallel}{1}} \leq 0$$

$$\begin{array}{c} x \in \mathrm{Conv}(X) \\ \shortparallel \\ x \in \mathrm{Conv}(\{x_1, \dots x_t\}) \end{array} \left\{ \begin{array}{l} x = \sum_{t=1}^{T} \lambda_t x_t \\ \sum_{t=1}^{T} \lambda_t = 1 \\ \lambda_t \geq 0 \qquad t = 1, \dots T \end{array} \right.$$

that is

$$\begin{aligned} w_{LD} = \max\ & c x \\ & D x \leq d \\ & x \in \mathrm{Conv}(X) \end{aligned}$$

called "Convexified relaxation"

So:

Theorem : $w_{LD} = \max\{c x : Dx \leq d, \ x \in \mathrm{Conv}(X)\}$

More generally this holds true for any $X = \{x \in \mathbb{Z}^n : Ax \leq b\}$

This theorem gives information on the strength of the Lagrangian dual. In certain cases, it is no stronger than the LP bound:

Corollary 1: if $X = \{x \in \mathbb{Z}^n : Ax \leq b\}$ and $conv(X) = \{x \in \mathbb{R}^n : Ax \leq b\}$, then $w_{LD} = \max\{cx : Dx \leq d, Ax \leq b\}$, which is the LP bound (LP relaxation)

Obs 1: since

$$(P_u) \quad z(u) = \max \; cx + u(d - Dx)$$
$$Ax \leq b$$
$$x \in \mathbb{Z}^n$$
,

then if (P_u) satisfies the integrality property, i.e. $conv(\{x \in \mathbb{Z}^n : Ax \leq b\}) = \{x \in \mathbb{R}^n : Ax \leq b\}$, then w_{LD} is equal to (and so no stronger than) the LP bound.

Obs 2: the property is true also for minimization and mixed integer problems.

example (UFL)

Recall that (P_u) has always an integer optimum solution; therefore w_{LD} corresponding to the considered Lagrangian relaxation is equal to the LP bound corresponding to (UFL_2).

Corollary 2: If (P) is a Linear Programming problem, then $w_{LD} = z$.

The proof of the Theorem suggests how to compute w_{LD}, i.e how to solve the Lagrangian Dual.

Corollary 3: $w_{LD} \leq z_{LP}$, where z_{LP} is the linear programming bound (for a maximization problem)

Proof: $conv(X) \subseteq \{x \in \mathbb{R}^n : Ax \leq b\}$ if $X = \{x \in \mathbb{Z}^n : Ax \leq b\}$.