Optimality, relaxations and bounds

(Wolsey : Chapter 2)

Given an optimization problem of form

$$z = \max_{x \in X} c(x)$$

how can we prove that a feasible x^* is optimal? If we find a <u>lower bound</u> (l.b.) $\underline{z} \leq z$ and an <u>upper bound</u> (u.b.) $\bar{z} \geq z$ such that $\underline{z} = c(x^*) = \bar{z}$, then x^* is optimal.

<u>Algorithmic consequence</u>:

- find a decreasing sequence of u.b. :

$$\bar{z}_1 > \bar{z}_2 > \dots > \bar{z}_s$$

- find an increasing sequence of l.b. :

$$\underline{z}_1 < \underline{z}_2 < \dots < \underline{z}_t$$

- stop when

$$\bar{z}_s - \underline{z}_t \leq \varepsilon$$

for a given $\varepsilon \geq 0$.

So : <u>how to compute l.b. and u.b.?</u>

<u>Primal bounds</u> ($\equiv$ l.b.)

each feasible solution $\bar{x} \in X$ gives a l.b. (for a maximization problem); in fact

$$c(\bar{x}) \leq z$$

* this is the way to compute l.b: *

<u>heuristics</u>

<u>Dual bounds</u> ($\equiv$ u.b.)

finding u.b. for a maximization problem (l.b. for a minimization problem) is not immediate:

* the most important approach is *

<u>"by relaxation"</u>

<u>Idea</u>: replace a "difficult" max (min) optimization problem by a "simpler" optimization problem s.t.:

<u>Def</u>: a problem (RP):

$$(RP) \qquad z^R = \max_{x \in T} f(x)$$

is a relaxation of problem (P):

$$(P) \qquad z = \max_{x \in X} c(x)$$

if:

(i) $X \subseteq T$

(ii) $f(x) \geq c(x) \quad \forall x \in X$

<u>Proposition</u>: if (RP) is a relaxation of (P), then $\boxed{z^R \geq z}$ (u.b.!)

<u>Proof</u>:

Assume x^* optimal solution to (P)

- $z = c(x^*) \leq f(x^*)$ from (ii)
- $f(x^*) \leq z^R$, since $x^* \in T$ from (i)

$\Rightarrow \quad z = c(x^*) \leq f(x^*) \leq z^R$ $\square$

How to construct useful relaxations?

1 LP relaxations

If (P) $\max cx$

$$
\begin{aligned}
Ax &\leq b \\
x &\geq 0 \\
x &\in \mathbb{Z}^n
\end{aligned}
$$

Integer Linear Program

then its linear programming relaxation is the linear program

$(\overline{P})$ $\max cx$

$$
\begin{aligned}
Ax &\leq b \\
x &\geq 0
\end{aligned}
$$

Just remove the integrality constraints

In fact: (i) holds true

(ii) " " since $f(x) = c(x)$

example

(P) $z = \max\ 4x_1 - x_2$

$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1, x_2 \geq 0, \text{ integer}$$

($\bar{P}$) $\bar{z} = \max\ 4x_1 - x_2$

$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Lower bound: Since $(2, 1)$ is feasible to (P), $4 \cdot 2 - 1 = 7$ is a l.b.:

$$z \geq 7$$

Upper bound: If we solve ($\bar{P}$) (via a LP solver) we get an optimal solution $x^* = \left(\frac{20}{7}, 3\right)$, with $\bar{z} = \frac{59}{7}$, which is an u.b.:

$$z \leq \frac{59}{7}$$

Since z must be integer, we can refine as

$$z \leq \left\lfloor \frac{59}{7} \right\rfloor = 8$$

So : $7 \leq z \leq 8$

Obs 1 : the definition of LP relaxation applies to (MILP) e to (BIP).

$$x \in \{0,1\} \Rightarrow \underset{\text{relax}}{0 \leq x \leq 1}$$

Obs 2 : the concept of "better" formulation is very important for LP relaxation:

Proposition If P_1 and P_2 are two polyhedra such that $P_1 \subset P_2$, and we consider the following alternative models :

1) $\max cx$

$x \in P_1$

x integer

2) $\max cx$

$x \in P_2$

x integer

alternative (and equivalent) models for a given problem

and consider the corresponding LP relaxations:

$$z^{LP_1} = \max cx \quad x \in P_1 \qquad\qquad z^{LP_2} = \max cx \quad x \in P_2$$

then $z^{LP_1} \leq z^{LP_2}$ for all c

□

<u>That is</u>: <u>better formulations give tighter</u> ($\leq$) dual <u>bounds</u> (upper bounds in the case of maximization)

In some cases general relaxations also allow to prove optimality:

Proposition:

(i) If a relaxation (RP) is infeasible, then the original problem (P) is infeasible

(ii) Let x^* an optimal solution of (RP). If $x^* \in X$ and $f(x^*) = c(x^*)$, then x^* is optimal for (P)

Proof:

(i) Since (RP) is infeasible, then $T = \phi$. Since $X \subseteq T$ then $X = \phi$, too.

(ii) Since $x^* \in X$:

- $z \geq c(x^*) \underset{\text{hyp.}}{=} f(x^*) = z^R$

- $z \leq z^R$ being a relaxation (of max)

$\Rightarrow z = z^R = c(x^*)$ □

example

$$(P) \quad z = \max \; 7x_1 + 4x_2 + 5x_3 + x_4$$

(Knapsack) $\quad 3x_1 + 3x_2 + 4x_3 + x_4 \leq 6$ ($= b$)

$$x \in \{0,1\}^4$$

The LP relaxation has optimal solution $x^* = (1,1,0,0)$.
It is integer (binary). So, it is optimal to the Knapsack problem.

On the Knapsack problem

- How to compute an optimal solution of the LP relaxation?

e.g. $\frac{7}{3} \geq \frac{4}{3} \geq \frac{5}{4} \geq 1$

i.e. order the items so that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}$$

then: $x_1^* = 1$, $x_2^* = 1$, $x_3^* = 0$, $x_4^* = 0$

binary solution

If the capacity ($\overset{b}{\shortparallel}$) would be 7?

$x_1^* = 1 \quad x_2^* = 1 \quad x_3^* = \frac{7-6}{4} = \frac{1}{4} \quad x_4^* = 0$

The optimal solution of the LP relaxation is $x^* = (1, 1, \frac{1}{4}, 0)$

$$c\,x^* = 7 + 4 + 5 \cdot \frac{1}{4} = \frac{49}{4} = 12 + \frac{1}{4}$$

is then an u.b. to z

- How to compute a feasible solution?

Greedy heuristic : follow the same item order

$\tilde{x}_1 = 1 \quad \tilde{x}_2 = 1 \quad \tilde{x}_3 = 0 \quad \tilde{x}_4 = 1$

so $\tilde{x} = (1, 1, 0, 1)$ is feasible, and

$c\,\tilde{x} = 7 + 4 + 1 = 12$ is a l.b. to z

So : $12 \leq z \leq 12 + \frac{1}{4}$

but $\lfloor 12 + \frac{1}{4} \rfloor = 12$ is also an u.b.

Therefore $\tilde{x}$ ($c\tilde{x} = 12$) is an optimal solution for the case $\boxed{b = 7}$

(2) Relaxations via the elimination of constraints

e.g. consider the link-path model (MCF_2), i.e.

$$\underset{Z_R}{\text{Min}} \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ij}^k \left[\sum_{p\in P^k} \delta_{ij}(p) f^k(p) \right] = \sum_{p\in P^k} c^k(p) \cdot f^k(p)$$

$$\sum_{p\in P^k} f^k(p) = d^k \qquad k=1,..K$$

$$\sum_{k=1}^{K} \sum_{p\in P^k} \delta_{ij}(p) f^k(p) \leq u_{ij} \quad \forall (i,j)\in A$$

$$f^k(p) \geq 0 \quad [\text{integer}] \quad \forall p\in P^k, \; k=1,..K$$

If we remove the capacity constraints, then we get a relaxation (why?)

* In this case the relaxation decomposes into K subproblem (one

per commodity) which are shortest path problems (easy!)
This is true also if $f(P)$ are required to be integer (ILP)

3 Lagrangian relaxation

An important extension of the idea is not just to drop "complicating" constraints (s.t. the capacity constraints in (MCF_2)), but to add them to the objective function via suitable multipliers.

Let (P)
$$z = \max cx$$
$$Ax \leq b$$
$$x \in X \quad \leftarrow \text{additional constraints}$$
$$x \in \mathbb{Z}^n$$

Given a vector $u \geq$ of Lagrange multipliers, a Lagrangian relaxation is

$$(P_u) \qquad z(u) = \max\ cx + u(b - Ax)$$
$$x \in X$$
$$x \in \mathbb{Z}^n$$

Proposition : $z(u) \geq z \qquad \forall\, u \geq 0$

Proof

Let x^* be an optimal solution to (P).

Therefore: $x^* \in X$, $x^* \in \mathbb{Z}^n$, $Ax^* \leq b$.

Since $u \geq 0$:

$$z = cx^* \leq cx^* + u(b - Ax^*) \leq z(u)$$

since x^* is feasible to (P_u)

$\square$

example (cont.)

a Lagrangian relaxation of (MCF_2) is

$$z(\mu) = \operatorname{Min} \sum_{p \in P^k} c^k(p) f^k(p) + \sum_{(i,j)\in A} \pi_{ij} \left(\sum_{k=1}^{K} \sum_{p \in P^k} \delta_{ij}(p) f^k(p) - u_{ij} \right)$$

$$\sum_{p \in P^k} f^k(p) = d^k \qquad k = 1, \dots K$$

$$f^k(p) \geq 0 \text{ [integer]} \quad \forall p \in P^k, \ k = 1, \dots K$$

- $\pi_{ij} \geq 0$ is the Lagrange multiplier of the capacity constraint related to (i,j), $\forall (i,j) \in A$

- we added "$(Ax - b)$" instead of "$(b - Ax)$" since this is a minimization problem

- again K shortest path problems

$$\boxed{z_R, \ z(\mu) \leq z} \rightarrow \text{l.b. !!}$$

In fact, the relaxation via elimination of constraints is a special Lagrangian relaxation : $u = 0$

<u>example</u> <u>Constrained Shortest Path</u> (<u>CSP</u>)

- $G = (N, A)$ directed graph
- c_{ij} (≥ 0) cost of (i,j), $\forall (i,j) \in A$
- l_{ij} (≥ 0) length of (i,j), " "
- s source
- t destination

L : given upper bound

<u>CSP</u> : find the minimum cost path from s to t among the paths with length $\leq L$

- (ILP)
- NP-Hard

Variables

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ belongs to the path} \\ 0 & \text{otherwise} \end{cases}$$

$\forall (i,j) \in A$

$$z = \min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\sum_{(i,j)\in FS(i)} x_{ij} - \sum_{(j,i)\in BS(i)} x_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad \sum_{(i,j)\in A} l_{ij} x_{ij} \leq L$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in A$$

- a relaxation by constraint elimination:
  drop $\Rightarrow$ z_R Shortest path

- a Lagrangian relaxation: drop $\Rightarrow$ from the set of constraints, and added it to the objective function:

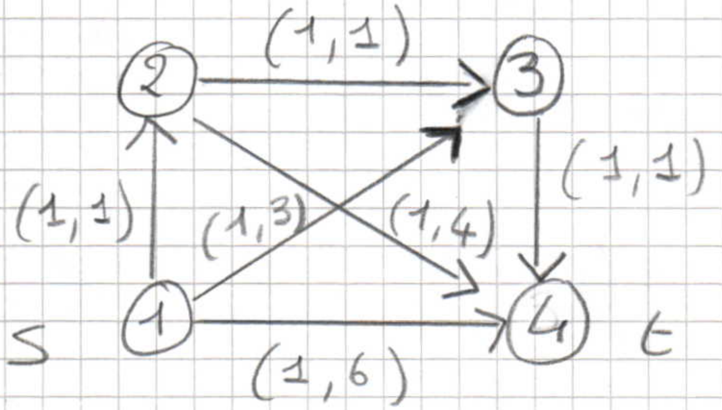
single multiplier

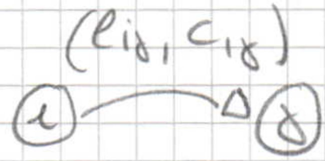
$$z(u) = \min_{(i,j)\in A} c_{ij} x_{ij} + u \Big(\sum_{(i,j)\in A} l_{ij} x_{ij} - L \Big) =$$

$$= -Lu + \min \sum_{(i,j)\in A} \underbrace{(c_{ij} + u \cdot l_{ij})}_{\text{new costs}} x_{ij}$$

Shortest path with modified costs

example





$L = 2$

- relaxation via constraint elimination:

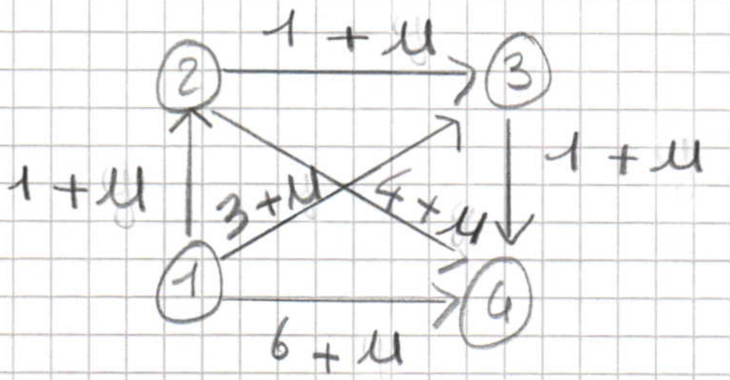
the optimal solution is the path

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$$

with cost 3 : $z_R = 3$

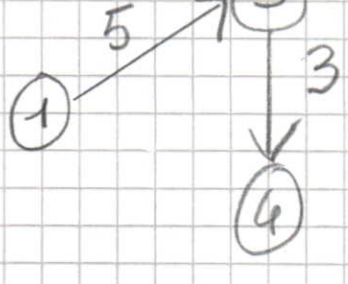
length 3 : infeasible to (CSP)

- general Lagrangian relaxation:



now the shortest path depend on u

If $u = 2$:



$$z(u) = z(2) = -2 \cdot u + 8 =$$
$$= -4 + 8 = 4$$

so $\quad z_R = 3 < z(2) = 4 \leq z \quad$ (in this example)

- How to choose the Lagrangian multiplier u?

- Note that $(1, 3, 4)$ is feasible to (CSP), and its "true" cost is 4 : therefore $(1, 3, 4)$ is an optimal solution to (CSP)

* In fact lower bound $z(u) = 4$ is equal to the upper bound 4 *