Basic Network Design Problems

(Pioro - Medhi : in [4.1, 4.2, 4.3, 4.4])

- Basic problems associated with normal (or nominal) state of communication networks
- All resources fully available

Notation

d : denotes demand between an origin and a destination ≡ also commodity

D : set of demands (commodities)

P_d : candidate paths to satisfy d

h_d : volume of demand of d, $\forall d \in D$

c_{ij} : unit cost of link (i,j), $\forall (i,j) \in A$

[$c_{ij} \leadsto \xi_e$ in P-M, with $e=(i,j)$]

Uncapacitated network design (4.1.1) (2)

Variables

- x_{dp} : flow on path $p \in P_d$, $\forall d \in D$, $p \in P_d$

< link-path formulation >

- y_{ij} : capacity to be associated with link (i,j), $\forall (i,j) \in A$

$$\text{Min} \sum_{(i,j) \in A} c_{ij} \, y_{ij} \quad \text{(bandwidth cost)}$$

$$\sum_{p \in P_d} x_{dp} = h_d \quad \forall d \in D \quad \text{(demand satisfaction)}$$

$$\sum_{d \in D} \sum_{p \in P_d} \delta_{ij}(dp) \, x_{dp} \leq y_{ij} \quad \forall (i,j) \in A \quad \text{(capacity constraints)}$$

(δ_{edp} in P-M)

$$x_{dp} \geq 0 \qquad \forall d \in D, \forall p \in P_d$$

where

$$\delta_{ij}(dp) = \begin{cases} 1 & \text{se } (i,j) \in p, \; p \in P_d \\ 0 & \text{otherwise} \end{cases}$$

Obs:

1) LP model: "easy", but not tractable in this form

2) in the optimal solution, y_{ij} are equal to the "load" of (i,j), $\forall (i,j) \in A$

3) indeed, an optimal solution can be obtained by sending h_d, $\forall d$, on a shortest path in P_d (w.r.t. $\{c_{ij}\}$):

shortest path allocation rule

easy!

4) the corresponding node-link formulation is (MCF1), by replacing u_{ij} by variables y_{ij}, $\forall (i,j) \in A$.

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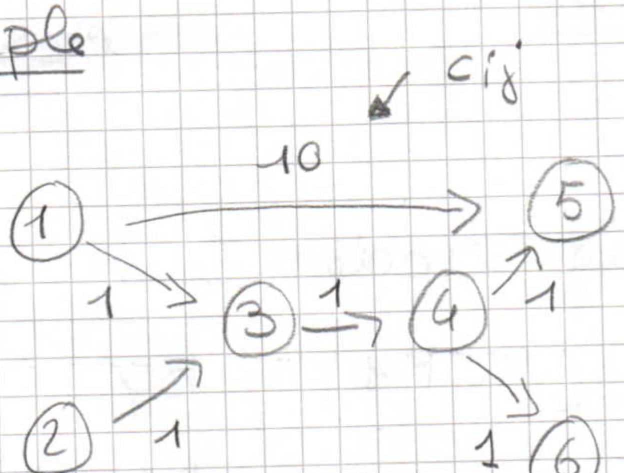
Capacitated version (4.1.2)

- when a capacity u_{ij} is given, $\forall (i,j) \in A$
- the model is obtained from (4.1.1) by substituting y_{ij} with u_{ij}, $\forall (i,j) \in A$

  (c_e in P-M)

<u>Obs</u>: no counterpart of the shortest path allocation rule is now available (due to capacities)

<u>example</u>



$u_{ij} = 2$
$\forall (i,j) \in A$

d_1 : pair $(1, 5)$; $h_{d_1} = 2$

d_2 : pair $(2, 6)$: $h_{d_2} = 2$

Possible objective functions ?

e.g. maximize the unused capacity:

$$\text{Max} \sum_{(i,j)\in A} r_{ij} \left(u_{ij} - \sum_{d\in D} \sum_{p\in P_d} \delta_{ij}(dp)\, x_{dp} \right)$$

where r_{ij} is the revenue associated with 1 unit of unused capacity of (i,j)

- linear function

- equivalent to:

$$\text{Min} \sum_{d\in D} \sum_{p\in P_d} r(dp) \cdot x_{dp}$$

where $r(dp) = \sum_{(i,j)\in A} r_{ij}\, \delta_{ij}(dp)$

* revenue associated with $p \in P_d$ *

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Routing restrictions

- Previous ND models : no routing restrictions
- Examples of routing restrictions

Path diversity (4.2.1)

Given a diversity factor n_d, $\forall d \in D$, force h_d to be split onto $\geq n_d$ different paths, with flow $\leq h_d/n_d$:

$$\sum_{p \in P_d} x_{dp} = h_d \qquad \forall d \in D$$

$$\sum_{d \in D} \sum_{p \in P_d} \delta_{ij}(dp)\, x_{dp} \leq u_{ij} \qquad \forall (i,j) \in A$$

$$x_{dp} \leq h_d/n_d \qquad \forall d \in D,\ p \in P_d$$

- Used to get routing solutions more robust to link failures

- Node-link counterpart: $x_{ij}^{d} \leq h_d/n_d \quad \forall (i,j) \in A, \ \forall d \in D$

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Lower bounds on nonzero flows (4.2.2)

At least l_d on each used path for commodity d:

Binary variables:

$$u_{dp} = \begin{cases} 1 & \text{if path } p \text{ is used to carry flow for demand } d \\ 0 & \text{otherwise} \end{cases} \qquad \forall d \in D,\ p \in P_d$$

$$\sum_{p \in P_d} x_{dp} = h_d \qquad \forall d \in D$$

$$\sum_{d \in D} \sum_{p \in P_d} \delta_{ij}(dp)\, x_{dp} \leq u_{ij} \qquad \forall (i,j) \in A$$

$$u_{dp} \cdot l_d \leq x_{dp} \leq u_{dp} \cdot h_d \qquad \forall d \in D,\ p \in P_d$$

$$u_{dp} \in \{0, 1\} \qquad \forall d \in D,\ p \in P_d$$

* From LP to ILP (= Integer Linear Programming) *

Limited demand split (4.2.3)

At most a predetermined number of used paths per demand (contrary to path diversity)

e.g. single path (single-path allocation, unsplittable routing, non-bifurcated routing)

$$x_{dp} = \boxed{h_d \cdot u_{dp}} \qquad \forall d \in D,\ p \in P_d$$

$$\sum_{p \in P_d} u_{dp} = 1 \qquad \forall d \in D \qquad \text{(path selector)}$$

$$\sum_{d \in D} \sum_{p \in P_d} \delta_{ij}(dp)\, x_{dp} \leq u_{ij} \qquad \forall (i,j) \in A$$

$$u_{dp} \in \{0, 1\} \qquad \forall d \in D,\ p \in P_d$$

NP-Hard

Note that variables $\{x_{dp}\}$ can be eliminated from the model

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notes

Further observation

1) **node - link formulations can be used to model unsplittable routing:**

$$1 \quad s_d \xrightarrow{1} \circ \xrightarrow{1} \quad \circ \xrightarrow{1} t_d \quad -1 \qquad \text{demand } d$$

just define flow variables $x_{ij}^{d} \in \{0, 1\}$, supply of s_d equal to 1, request of t_d equal to -1

In this way, we can limit the number of links ($\equiv$ hops) in each path:

$$\sum_{(i,j) \in A} x_{ij}^{d} \leq \alpha \qquad \forall d \in D$$

(no more than α links)

2) we may want integral flows:

- $x_{dp} \geq 0$, integer $\forall d \in D$, $p \in P_d$

or

- $x_{ij}^{d} \geq 0$, integer $\forall (i,j) \in A$, $d \in D$

NP - Hard

Budget constraint (4.4)

- Substitute the explicit minimization of the cost function
- Allows the introduction of another objective function, e.g. <u>throughput maximization</u>

$$\text{Max } r$$

$$\sum_{p \in P_d} x_{dp} \geqslant r \cdot h_d \quad \forall d \in D$$

$$\sum_{d \in D} \sum_{p \in P_d} \delta_{ij}(dp)\, x_{dp} \leqslant y_{ij} \quad \forall (i,j) \in A$$

$$\boxed{\sum_{(i,j) \in A} c_{ij} \cdot y_{ij} \leqslant B} \quad \text{budget constraint}$$

$$x_{dp} \geqslant 0 \quad \forall d \in D, \forall p \in P_d$$

r : proportion of the realized demand volumes

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Nonlinear link dimensioning

- Influence of other types of link dimensioning functions (besides the assumed linear functions)

1. Modular links (6.3.1)

- common feature in communication networks

- let M the size of the link capacity module

The uncapacitated network design model (in 6.1.1) modifies as follows:

Variables:

y_{ij}: number of modules for link (i,j) , $\forall (i,j) \in A$

< from continuous to integer >

$$\text{Min} \sum_{(i,j)\in A} c_{ij} \cdot y_{ij}$$

$$\sum_{p\in P_d} x_{dp} = h_d \qquad \forall d \in D$$

$$\sum_{d\in D} \sum_{p\in P_d} \delta_{ij}(dp)\, x_{dp} \le M \cdot y_{ij} \qquad \forall (i,j)\in A$$

$$x_{dp} \geq 0 \qquad \forall d\in D, \forall p \in P_d$$

$$y_{ij} \geq 0 \text{ , integer} \qquad \forall (i,j)\in A$$

where now c_{ij} is the cost of a module

NP-Hard

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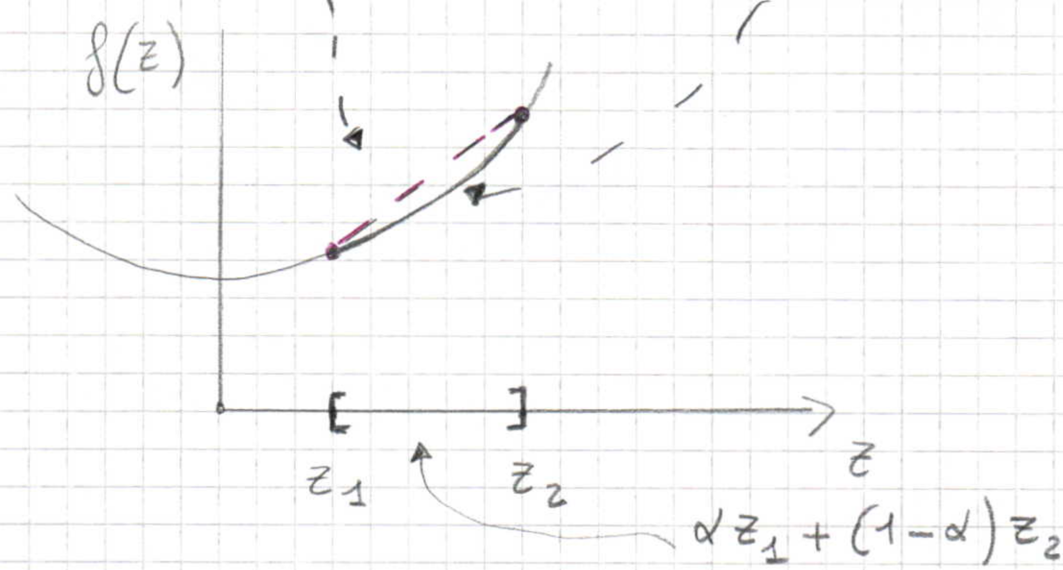
2. Convex cost and delay functions

(4.3.2)

<u>Def</u> : a function $f : \mathbb{R} \to \mathbb{R}$ is <u>convex</u> if and only if, for each two points z_1 and z_2 in $\mathbb{R}$ and each $\alpha \in [0,1]$:

$$\alpha \cdot f(z_1) + (1-\alpha) f(z_2) \geq f(\alpha z_1 + (1-\alpha) z_2)$$

e.g.



Typically convex cost functions are used in communication networks to model <u>delay</u>, e.g:

$$F_{ij}(x_{ij}) = \frac{1}{(\mu_{ij} - x_{ij})}, \quad 0 \leq x_{ij} < \mu_{ij}$$

where $x_{ij} = \sum_{d \in D} \sum_{p \in \mathcal{P}_d} x_{dp}$ is the <u>load</u> of (i,j)

(CXP)

Typical convex programming model

to minimize the delay:

$$\min \sum_{(i,\gamma)\in A} F_{i\gamma}(x_{i\gamma})$$

* the sum of convex functions is convex *

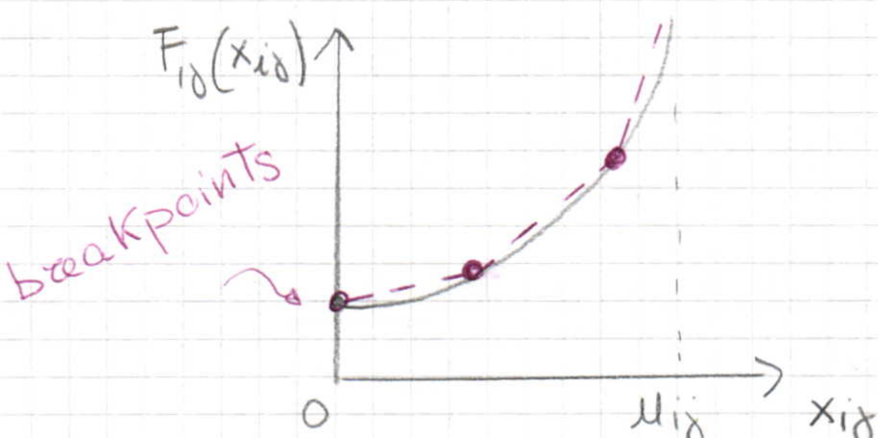
$$\sum_{p\in \mathcal{P}_d} x_{dp} = h_d \quad \forall d \in D$$

$$\sum_{d\in D} \sum_{p\in \mathcal{P}_d} \delta_{i\gamma}(dp)\, x_{dp} = x_{i\gamma} \quad \forall (i,\gamma)\in A$$

load definition

$$x_{i\gamma} \leq u_{i\gamma} \quad \forall (i,\gamma)\in A$$

- (CXP) is "easy": no local minima but only global minima
- Often, it is approximated via a piecewise linear approximation, and solved via LP (due to convexity)



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In fact, if we use K_{ij} linear functions $b^k_{ij} + a^k_{ij} \cdot x_{ij}$, $k = 1, .. K_{ij}$, to approximate $F_{ij}(x_{ij})$, we get:

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$$\text{Min} \sum_{(i,j) \in A} r_{ij} \qquad (LP)$$

$$\sum_{p \in P_d} x_{dp} = h_d \qquad \forall d \in D$$

$$\sum_{d \in D} \sum_{p \in P_d} \delta_{ij}(d_p)\, x_{dp} = x_{ij} \qquad \forall (i,j) \in A$$

$$x_{ij} \leq u_{ij} \qquad \forall (i,j) \in A$$

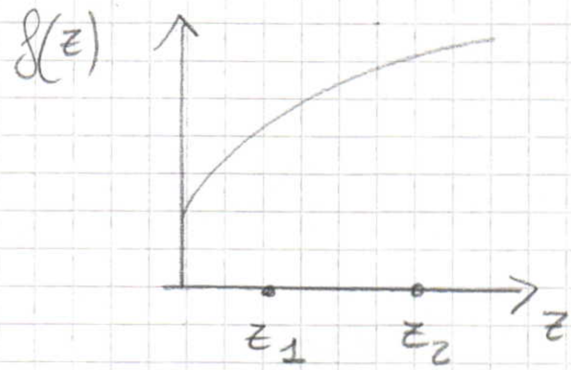
$$\boxed{r_{ij} \geq b^k_{ij} + a^k_{ij}\, x_{ij} \qquad \forall (i,j) \in A,\ k = 1, .. K_{ij}}$$

<u>In fact</u>: the piecewise linear approximation is the maximum of the K_{ij} linear functions; the auxiliary variable r_{ij}, $\forall (i,j) \in A$, gives an upper bound; by minimizing r_{ij} (indeed, $\sum_{(i,j) \in A} r_{ij}$), we then minimize the piecewise linear approximation!

3. Concave link dimensioning

(4.3.3)

- A function f is concave if and only if $-f$ is convex
- Often, link dimensioning functions are concave rather than convex:



$\leftarrow \dfrac{f(z_1)}{z_1} \geq \dfrac{f(z_2)}{z_2}$ for $z_1 < z_2$

is in fact quite natural...

However: the minimization of a concave function is typically NP-Hard (numerous local minima!)

Therefore: LP approximation (via piecewise linear functions) can not be achieved as shown for the convex case

I.L.P. is required!