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PSC 2023/24 (375AA, 9CFU)

Principles for Software Composition

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25b - CTMC

Probability law

Cumulative distribution function (probability law)

$$X : \Omega \to \mathbb{R}$$

$$F_X(x) \triangleq P(X \leq x) = P(\{\omega \in \Omega \mid X(\omega) \leq x\})$$

$$P(X \leq a) = F_X(a)$$

$$P(X > a) = 1 - F_X(a)$$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

Probability density

$$X : \Omega \to \mathbb{R}$$

integrable $f_X : \mathbb{R} \to [0, +\infty)$

such that $F_X(a) = \int_{-\infty}^{a} f_X(x)\ dx$

i.e. $P(a < X \leq b) = \int_{a}^{b} f_X(x)\ dx$

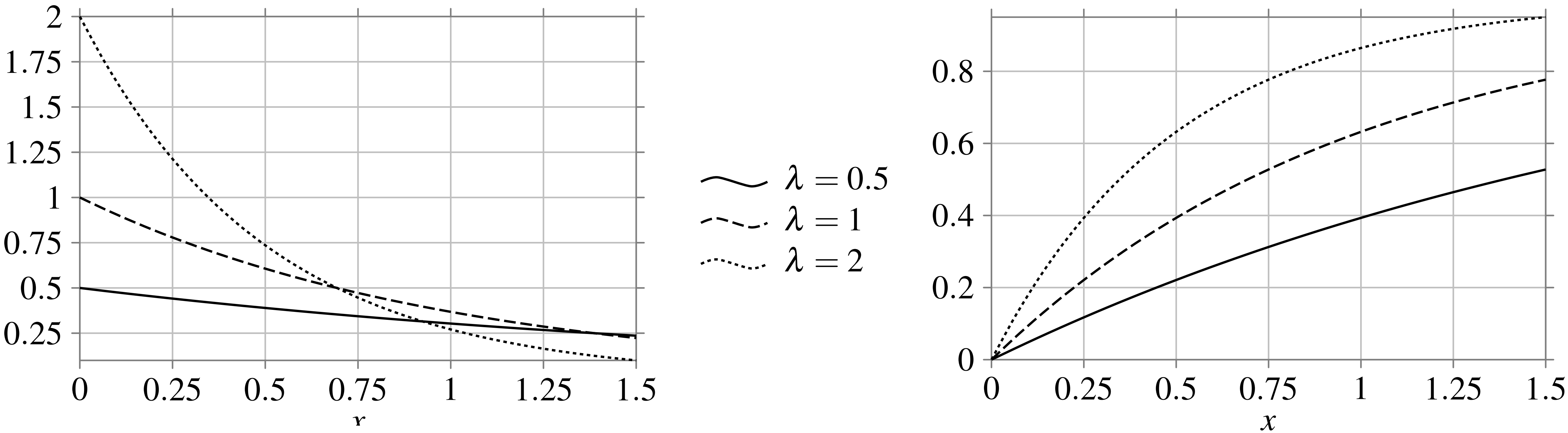
note that $P(X = a)$ is usually 0 when X is continuous

Exponential distribution

(Negative) Exp distribution

rate λ

$$f_X(x) \triangleq \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad F_X(x) \triangleq \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



expected value (mean) $1/\lambda$ variance $1/\lambda^2$

Properties: memoryless

memoryless $P(X > a + b | X > a) = P(X > b)$

recalling that $P(X > a) = e^{-\lambda a}$

$$P(X > a + b \,|\, X > a) = \frac{P(X > a + b \wedge X > a)}{P(X > a)}$$

$$= \frac{P(X > a + b)}{P(X > a)} = \frac{\lambda e^{-\lambda(a+b)}}{\lambda e^{-\lambda a}} = e^{-\lambda b} = P(X > b)$$

Properties: memoryless

memoryless $P(X > a + b | X > a) = P(X > b)$

it is the only memoryless distribution!

$$\frac{P(X > a + b)}{P(X > a)} = P(X > b)$$

$$P(X > a + b) = P(X > a)\ P(X > b)$$

$$P(X > a) = P(X > 1)^a = e^{a \ln P(X>1)} = e^{-\lambda a}$$

where $\lambda = -\ln P(X > 1)$

Properties: min

(X_1, λ_1) (X_2, λ_2) (independent, exponentially distributed)

$X(\omega) = \min\{X_1(\omega), X_2(\omega)\}$ is exponentially distributed

$(X, \lambda_1 + \lambda_2)$

$P(X \leq x) \triangleq 1 - e^{-(\lambda_1+\lambda_2)x}$

$$
\begin{aligned}
& P(\min\{X_1, X_2\} > x) \\
&= P(X_1 > x \wedge X_2 > x) \\
&= P(X_1 > x)\; P(X_2 > x) \\
&= e^{-\lambda_1 x} e^{-\lambda_2 x} \\
&= e^{-(\lambda_1+\lambda_2)x}
\end{aligned}
$$

(related to sojourn time in CTMC)

(exploited in PEPA)

Properties: precedence

(X_1, λ_1) (X_2, λ_2) (independent, exponentially distributed)

$$P(X_1 < X_2) \triangleq \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$\begin{aligned}
P(X_1 < X_2) &= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} \left(\int_{t_1}^\infty \lambda_2 e^{-\lambda_2 t_2} dt_2 \right) dt_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} \left[e^{-\lambda_2 t_2} \right]_\infty^{t_1} dt_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} \cdot e^{-\lambda_2 t_1} dt_1 \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) t_1} dt_1 \\
&= \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2) t} \right]_\infty^0 \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

(related to embedded DTMC)

(exploited in PEPA)

(homogeneous) CTMC

CTMC

$(\Omega, \mathcal{A}, P)$ probability space

$\{X_t\}_{t\in\mathbb{R}}$ homogeneous Markov chain

$$P(X_{t_n+\Delta t} = x | X_{t_n} = x_n, \cdots, X_{t_0} = x_0) = P(X_{\Delta t} = x | X_0 = x_n)$$

let T_i be the time spent in state i
before making a transitions to any other state

$$P(T_i > t + \Delta t | T_i > t) = P(T_i > \Delta t)$$

$$P(T_i > 15 | T_i > 10) = P(T_i > 5)$$

the random variable T_i is memoryless

it must be exponentially distributed

rate λ_i, mean $1/\lambda_i$

CTMC

$(\Omega, \mathcal{A}, P)$ probability space

$\{X_t\}_{t\in\mathbb{R}}$ homogeneous Markov chain

$$P(X_{t_n+\Delta t} = x | X_{t_n} = x_n, \cdots, X_{t_0} = x_0) = P(X_{\Delta t} = x | X_0 = x_n)$$

let $T_{i,j}$ be the time spent in state i before making a transitions to state j

$$P(T_{i,j} > t + \Delta t | T_{i,j} > t) = P(T_{i,j} > \Delta t)$$

the random variable $T_{i,j}$ is memoryless

it must be exponentially distributed

rate $\lambda_{i,j}$, mean $1/\lambda_{i,j}$

$$T_i = \min_{j\neq i} T_{i,j} \qquad \lambda_i = \sum_{j\neq i} \lambda_{i,j}$$

Embedded DTMC

$(\Omega, \mathcal{A}, P)$ probability space $\{X_t\}_{t \in \mathbb{R}}$ homogeneous Markov chain

$$T_i = \min_{j \neq i} \; T_{i,j} \qquad \lambda_i = \sum_{j \neq i} \lambda_{i,j}$$

$$p_{i,j} = P\left(\bigwedge_{k \neq i,j} T_{i,j} < T_{i,k} \right) = P\left(T_{i,j} < \min_{k \neq i,j} T_{i,k} \right) = \frac{\lambda_{i,j}}{\lambda_i}$$

$$P = \begin{bmatrix} 0 & p_{1,2} & \cdots & p_{1,N} \\ p_{2,1} & 0 & \cdots & p_{2,N} \\ \vdots & \vdots & & \vdots \\ p_{N,1} & p_{N,2} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_{1,2}/\lambda_1 & \cdots & \lambda_{1,N}/\lambda_1 \\ \lambda_{2,1}/\lambda_2 & 0 & \cdots & \lambda_{2,N}/\lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_{N,1}/\lambda_N & \lambda_{N,2}/\lambda_N & \cdots & 0 \end{bmatrix}$$

Embedded DTMC

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$$\begin{cases} \pi = \pi \cdot P \\ \sum_{i=1}^{N} \pi_i = 1 \end{cases}$$

(if ergodic) we get a steady state distribution

but the embedded DTMC ignores
the amount of time spent in each state

Balance equations

$(\Omega, \mathcal{A}, P)$ probability space $\{X_t\}_{t \in \mathbb{R}}$ homogeneous Markov chain

let p_i be the long time proportion of time spent in state i with respect to the time spent in other states

the flow in/out of each state i must balance

$\lambda_i p_i$ outgoing flow

rate at which transitions out of state i occur

$\lambda_{k,i} p_k$

rate at which transitions into state i occur from state k

$\sum_{k \neq i} \lambda_{k,i} p_k$ incoming flow

rate at which transitions into state i occur from other states

Infinitesimal matrix gen

$(\Omega, \mathcal{A}, P)$ probability space $\{X_t\}_{t \in \mathbb{R}}$ homogeneous Markov chain

outgoing flow $\lambda_i p_i = \sum_{k \neq i} \lambda_{k,i} p_k$ incoming flow

$$\lambda_i = \sum_{j \neq i} \lambda_{i,j}$$

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_{1,2} & \cdots & \lambda_{1,N} \\ \lambda_{2,1} & -\lambda_2 & \cdots & \lambda_{2,N} \\ \vdots & \vdots & & \vdots \\ \lambda_{N,1} & \lambda_{N,2} & \cdots & -\lambda_N \end{bmatrix}$$

$$\begin{cases} p \cdot Q = 0 \\ \sum_{i=1}^{N} p_i = 1 \end{cases}$$

Stationary distributions

$$P = \begin{bmatrix} 0 & p_{1,2} & \cdots & p_{1,N} \\ p_{2,1} & 0 & \cdots & p_{2,N} \\ \vdots & \vdots & & \vdots \\ p_{N,1} & p_{N,2} & \cdots & 0 \end{bmatrix}$$

$$\begin{cases} \pi = \pi \cdot P \\ \sum_{i=1}^{N} \pi_i = 1 \end{cases}$$

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_{1,2} & \cdots & \lambda_{1,N} \\ \lambda_{2,1} & -\lambda_2 & \cdots & \lambda_{2,N} \\ \vdots & \vdots & & \vdots \\ \lambda_{N,1} & \lambda_{N,2} & \cdots & -\lambda_N \end{bmatrix}$$

$$\begin{cases} p \cdot Q = 0 \\ \sum_{i=1}^{N} p_i = 1 \end{cases}$$

π_i proportion of transitions into state i

$1/\lambda_i$ mean time spent into state i

if $\forall i, j.\ \lambda_i = \lambda_j$ then $\forall i.\ p_i = \pi_i$

$$p_i = \frac{\pi_i/\lambda_i}{\sum_j \pi_j/\lambda_j}$$

Example

A server can serve up to two requests

one/two new requests arrive with rate λ

one/two requests are served with rate μ

represent the system as a CTMC, by defining its

infinitesimal generator matrix

embedded DTMC

find the steady state distribution when $\lambda = \mu$

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$$Q = \left[\qquad \right] \quad P = \left[\qquad \right]$$

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represent the system as a CTMC, by defining its

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embedded DTMC

find the steady state distribution when $\lambda = \mu$

$$Q = \begin{bmatrix} -2\lambda & \lambda & \lambda \\ \mu & -\lambda - \mu & \lambda \\ \mu & \mu & -2\mu \end{bmatrix} P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ \mu/\lambda + \mu & 0 & \lambda/\lambda + \mu \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Example

find the steady state distribution when $\lambda = \mu$

$$Q = \begin{bmatrix} -2\lambda & \lambda & \lambda \\ \mu & -\lambda-\mu & \lambda \\ \mu & \mu & -2\mu \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ \mu/\lambda+\mu & 0 & \lambda/\lambda+\mu \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Example

find the steady state distribution when $\lambda = \mu$

$$Q = \begin{bmatrix} -2\lambda & \lambda & \lambda \\ \lambda & -2\lambda & \lambda \\ \lambda & \lambda & -2\lambda \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\begin{cases} -2\lambda p_1 + \lambda p_2 + \lambda p_3 = 0 \\ \lambda p_1 - 2\lambda p_2 + \lambda p_3 = 0 \\ \lambda p_1 + \lambda p_2 - 2\lambda p_3 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases} \qquad \begin{cases} \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 = \pi_1 \\ \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 = \pi_2 \\ \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$p = [\ 1/3\ ,\ 1/3\ ,\ 1/3\ ] \qquad \pi = [\ 1/3\ ,\ 1/3\ ,\ 1/3\ ]$$

CTMC as LTS

$$\alpha_C : S \to S \to \mathbb{R} \qquad\qquad \alpha_C\ i\ j = \lambda_{i,j}$$

embedded DTMC

$$\alpha_D : S \to \mathbb{D}(S) \qquad \alpha_D\ i\ j = \begin{cases} \frac{\lambda_{i,j}}{\sum_{k \neq i} \lambda_{i,k}} & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

can we derive a notion of equivalence between states?