

PSC 2023/24 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni

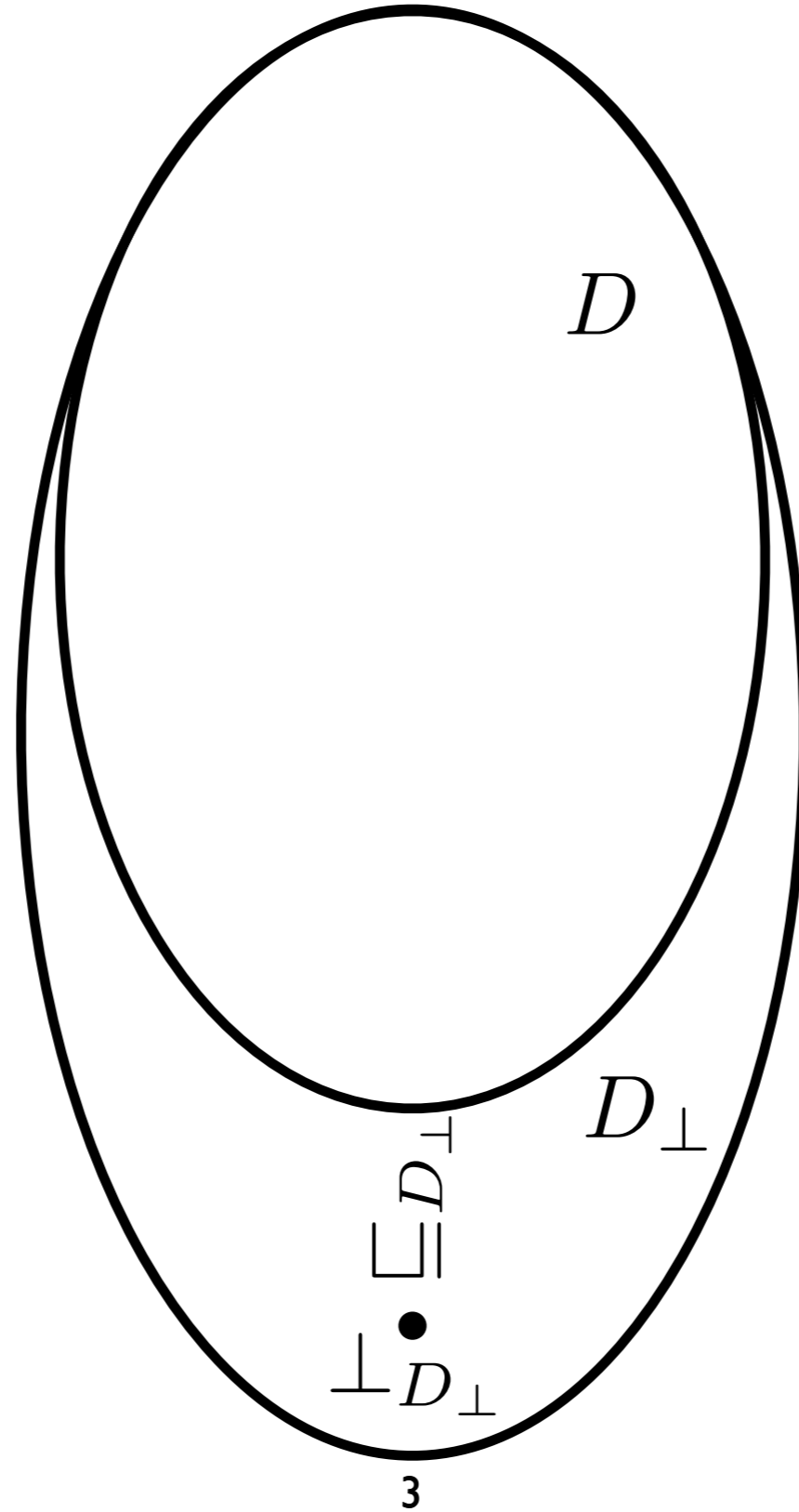
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13c - Continuity theorems

Lifted Domains

Lifted Domains



Lifted Domains

$$\mathcal{D} = (D, \sqsubseteq_D) \text{ CPO} \quad \Rightarrow \quad \mathcal{D}_\perp = (D_\perp, \sqsubseteq_{D_\perp})$$

$$\begin{aligned} D_\perp &\triangleq \{\perp\} \uplus D \\ &= \{(0, \perp)\} \cup (\{1\} \times D) = \{(0, \perp)\} \cup \{(1, d) \mid d \in D\} \end{aligned}$$

$$\begin{aligned} \perp_{D_\perp} &\triangleq (0, \perp) & [\cdot] : D &\rightarrow D_\perp & \text{lifting function} \\ [d] &\triangleq (1, d) \end{aligned}$$

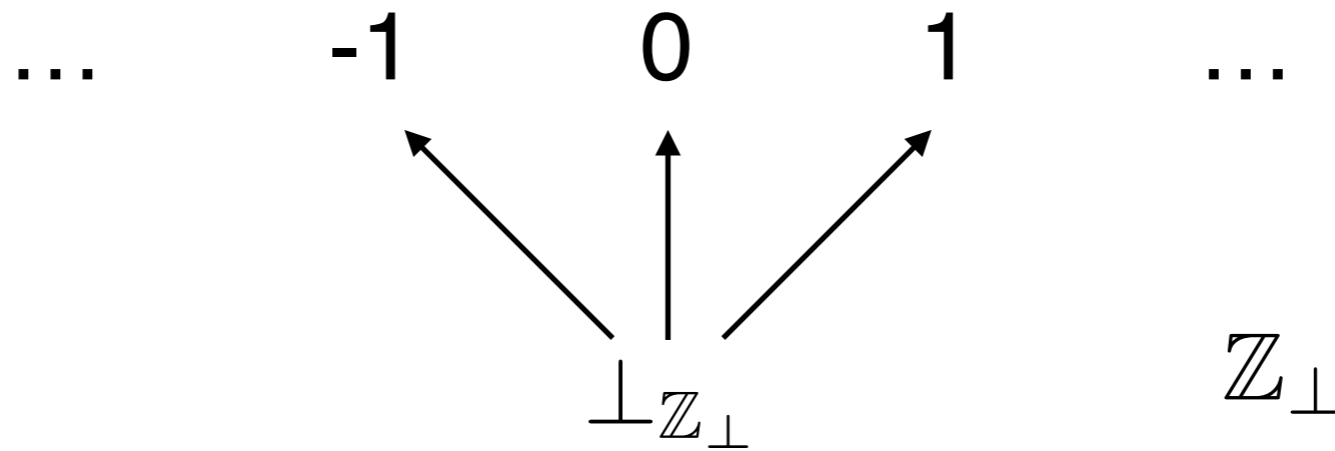
how to order lifted elements?

$$\forall x \in D_\perp. \perp_{D_\perp} \sqsubseteq_{D_\perp} x$$

$$\forall d_1, d_2 \in D. [d_1] \sqsubseteq_{D_\perp} [d_2] \Leftrightarrow d_1 \sqsubseteq_D d_2$$

Example

$(\mathbb{Z}, =)$



Lifted Domains

TH. $\mathcal{D}_\perp = (D_\perp , \sqsubseteq_{D_\perp}) \text{ CPO}_\perp$

try on your own to prove:

PO,

bottom element,

complete

observe that: $\bigsqcup_{i \in \mathbb{N}} \lfloor d_i \rfloor = \left\lfloor \bigsqcup_{i \in \mathbb{N}} d_i \right\rfloor$

it is an upper bound

it is the least upper bound

Lifting operator

(D, \sqsubseteq_D) CPO

(E, \sqsubseteq_E) CPO $_{\perp}$

$$(\cdot)^* : [D \rightarrow E] \rightarrow [D_{\perp} \rightarrow E]$$

$$\forall f \in [D \rightarrow E]. f^*(x) \triangleq \begin{cases} \perp_E & \text{if } x = \perp_{D_{\perp}} \\ f(d) & \text{if } x = [d] \end{cases}$$

for the definition to be well-given
we need to prove:

$$f \in [D \rightarrow E] \quad \Rightarrow \quad f^* \in [D_{\perp} \rightarrow E]$$

f continuous implies f^* continuous

TH. the lifting operator is well-defined

proof. assume f continuous, take a chain $\{x_n\}_{n \in \mathbb{N}}$ in D_\perp

we need to prove $f^* \left(\bigsqcup_{n \in \mathbb{N}} x_n \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_n)$

if $\forall n \in \mathbb{N}. x_n = \perp_{D_\perp}$ then it is obvious

otherwise, let $k = \min\{i \mid x_i \neq \perp_{D_\perp}\}$

then $\forall m \geq k. \exists d_m \in D. x_m = \lfloor d_m \rfloor$

and by prefix independence of lub

$$\bigsqcup_{n \in \mathbb{N}} x_n = \bigsqcup_{n \in \mathbb{N}} x_{n+k} \quad \bigsqcup_{n \in \mathbb{N}} f^*(x_n) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k})$$

we can just prove $f^* \left(\bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k})$

(see next slide)

(continue)

$$f^* \left(\bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k})$$

$$f^* \left(\bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = f^* \left(\bigsqcup_{n \in \mathbb{N}} \lfloor d_{n+k} \rfloor \right)$$

by def of k

$$= f^* \left(\left[\bigsqcup_{n \in \mathbb{N}} d_{n+k} \right] \right)$$

by lub in a lifted domain

$$= f \left(\bigsqcup_{n \in \mathbb{N}} d_{n+k} \right)$$

by def of lifting

$$= \bigsqcup_{n \in \mathbb{N}} f(d_{n+k})$$

by continuity of f

$$= \bigsqcup_{n \in \mathbb{N}} f^*(\lfloor d_{n+k} \rfloor)$$

by def of lifting

$$= \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k})$$

by def of k

TH. $(\cdot)^*$ is monotone

(try to prove on your own)

TH. $(\cdot)^*$ is continuous

proof. take a chain of continuous functions $\{f_i : D \rightarrow E\}_{i \in \mathbb{N}}$

we need to prove
$$\left(\bigsqcup_{i \in \mathbb{N}} f_i \right)^* = \bigsqcup_{i \in \mathbb{N}} f_i^*$$

take a generic $x \in D_\perp$

we need to prove
$$\left(\bigsqcup_{i \in \mathbb{N}} f_i \right)^* (x) = \left(\bigsqcup_{i \in \mathbb{N}} f_i^* \right) (x)$$

if $x = \perp_{D_\perp}$ it is obvious

if $x = \lfloor d \rfloor$ we have...

(see next slide)

(continue)

$$\left(\bigsqcup_{i \in \mathbb{N}} f_i \right)^* (\llbracket d \rrbracket) = \left(\bigsqcup_{i \in \mathbb{N}} f_i^* \right) (\llbracket d \rrbracket)$$

$$\left(\bigsqcup_{i \in \mathbb{N}} f_i \right)^* (\llbracket d \rrbracket) = \left(\bigsqcup_{i \in \mathbb{N}} f_i \right) (d) \quad \text{by def of lifting}$$

$$= \bigsqcup_{i \in \mathbb{N}} f_i(d) \quad \text{by lub in a functional domain}$$

$$= \bigsqcup_{i \in \mathbb{N}} f_i^*(\llbracket d \rrbracket) \quad \text{by def of lifting}$$

$$= \left(\bigsqcup_{i \in \mathbb{N}} f_i^* \right) (\llbracket d \rrbracket) \quad \text{by lub in a functional domain}$$

Let notation (de-lifting)

$$(E, \sqsubseteq_E) \text{ CPO}_\perp \quad \lambda x. e \in [D \rightarrow E] \quad t \in D_\perp$$

$$\text{let } x \leftarrow t. e \triangleq \underbrace{\underbrace{\underbrace{(\lambda x. e)^*}_{[D \rightarrow E]} \underbrace{t}_{D_\perp}}_{[D_\perp \rightarrow E]}}_E = \begin{cases} \perp_E & \text{if } t = \perp_{D_\perp} \\ e^{[d/x]} & \text{if } t = [d] \end{cases}$$

intuitively:

if t is a lifted value $[d]$ then we de-lift the value and
assign it to x in e

otherwise returns \perp_E

Continuity theorems

TH. (D, \sqsubseteq_D) CPO $f : D \rightarrow E_1 \times E_2$ $g_i \triangleq \pi_i \circ f$
 (E_i, \sqsubseteq_{E_i})
 f is continuous iff g_1, g_2 are continuous

proof. \Rightarrow) f is continuous $\Rightarrow g_i$ is continuous
 π_i is continuous

\Leftarrow) note that $\forall d \in D. f(d) = (g_1(d), g_2(d))$

assume g_1, g_2 are continuous

we want to prove f is continuous

take a chain $\{d_i\}_{i \in \mathbb{N}}$ in D

we must prove $f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

(see next slide)

(continue)

$$f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$$

$$f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \left(g_1 \left(\bigsqcup_{i \in \mathbb{N}} d_i \right), g_2 \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) \right) \quad \text{by def } g_1, g_2$$

$$= \left(\bigsqcup_{i \in \mathbb{N}} g_1(d_i), \bigsqcup_{i \in \mathbb{N}} g_2(d_i) \right) \quad g_1, g_2 \text{ are continuous}$$

$$= \bigsqcup_{i \in \mathbb{N}} (g_1(d_i), g_2(d_i)) \quad \text{by def of lub of pairs}$$

$$= \bigsqcup_{i \in \mathbb{N}} f(d_i) \quad \text{by def } g_1, g_2$$

TH. (D, \sqsubseteq_D)
 (E, \sqsubseteq_E) CPO $f : D \times E \rightarrow F$
 (F, \sqsubseteq_F)

$$f_d : E \rightarrow F$$

$$f_d \triangleq \lambda e. f(d, e)$$

$$f_e : D \rightarrow F$$

$$f_e \triangleq \lambda d. f(d, e)$$

f is continuous iff $\forall d \in D. f_d$ are continuous
 $\forall e \in E. f_e$ are continuous

proof. \Rightarrow) assume f is continuous

take a generic $d \in D$

we want to prove f_d is continuous

take a chain $\{e_i\}_{i \in \mathbb{N}}$ in E

$$\text{we prove } f_d \left(\bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)$$

(see next slide)

$$e \in E$$

$$f_e$$

(omitted)

(continue)

$$f_d \left(\bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)$$

$$f_d \left(\bigsqcup_{i \in \mathbb{N}} e_i \right) = f \left(d, \bigsqcup_{i \in \mathbb{N}} e_i \right) \quad \text{by def of } f_d$$

$$= f \left(\bigsqcup_{i \in \mathbb{N}} d, \bigsqcup_{i \in \mathbb{N}} e_i \right) \quad \text{by lub of constant chain}$$

$$= f \left(\bigsqcup_{i \in \mathbb{N}} (d, e_i) \right) \quad \text{by lub of pairs}$$

$$= \bigsqcup_{i \in \mathbb{N}} f(d, e_i) \quad \text{by continuity of } f$$

$$= \bigsqcup_{i \in \mathbb{N}} f_d(e_i) \quad \text{by def of } f_d$$

TH.

(D, \sqsubseteq_D)

(E, \sqsubseteq_E)

(F, \sqsubseteq_F)

CPO

$f : D \times E \rightarrow F$

$f_d : E \rightarrow F$

$f_d \triangleq \lambda e. f(d, e)$

$f_e : D \rightarrow F$

$f_e \triangleq \lambda d. f(d, e)$

f is continuous

iff

$\forall d \in D. f_d$ are continuous

$\forall e \in E. f_e$ are continuous

\Leftarrow) assume f_d, f_e are continuous for all d, e

we want to prove f is continuous

take a chain $\{(d_k, e_k)\}_{k \in \mathbb{N}}$ in $D \times E$

we prove $f \left(\bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)$

(see next slide)

(continue)

$$f \left(\bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)$$

$$f(\bigsqcup_k (d_k, e_k)) = f(\bigsqcup_i d_i, \bigsqcup_j e_j) \quad \text{by def of lub of pairs}$$

$$= f_d(\bigsqcup_j e_j) \quad \text{by def of } f_d \text{ with } d \triangleq \bigsqcup_i d_i$$

$$= \bigsqcup_j f_d(e_j) \quad \text{by continuity of } f_d$$

$$= \bigsqcup_j f(d, e_j) \quad \text{by def of } f_d$$

$$= \bigsqcup_j f_{e_j}(d) \quad \text{by def of } f_{e_j}$$


$$= \bigsqcup_j f_{e_j}(\bigsqcup_i d_i) \quad \text{by def of } d \triangleq \bigsqcup_i d_i$$

$$= \bigsqcup_j \bigsqcup_i f_{e_j}(d_i) \quad \text{by continuity of } f_{e_j}$$

$$= \bigsqcup_j \bigsqcup_i f(d_i, e_j) \quad \text{by def of } f_{e_j}$$

$$= \bigsqcup_k f(d_k, e_k) \quad \text{by switch lemma (applicable?)}$$

(continue) $f \left(\bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)$

if $i \leq n \wedge j \leq m$ then $f(d_i, e_j) \sqsubseteq f(d_n, e_m)$? 

\Downarrow

$$d_i \sqsubseteq_D d_n \wedge e_j \sqsubseteq_E e_m$$

$$f(d_i, e_j) = f_{d_i}(e_j) \sqsubseteq f_{d_i}(e_m) = f(d_i, e_m) = f_{e_m}(d_i) \sqsubseteq f_{e_m}(d_n) = f(d_n, e_m)$$

f_{d_i}

monotone

f_{e_m}

monotone

$$= \bigsqcup_j \bigsqcup_i f(d_i, e_j) \quad \img alt="green checkmark" data-bbox="918 765 961 822"/>$$

$$= \bigsqcup_k f(d_k, e_k) \quad \text{by switch lemma (applicable?)}$$

Some important functions

Apply

(D, \sqsubseteq_D)
 (E, \sqsubseteq_E) CPO

$apply : [D \rightarrow E] \times D \rightarrow E$
 $apply(f, d) \triangleq f(d)$

TH. $apply$ is monotone

(try to prove on your own)

TH. $apply$ is continuous

proof. from a previous theorem, we prove continuity
on each parameter separately $apply_f$ $apply_d$

1. for any $f \in [D \rightarrow E]$ $apply_f \triangleq \lambda d. f(d)$ is continuous

2. for any $d \in D$ $apply_d \triangleq \lambda f. f(d)$ is continuous

(see next slides)

1. for any $f \in [D \rightarrow E]$ $apply_f \triangleq \lambda d. f(d)$ is continuous

take $f \in [D \rightarrow E]$ and a chain $\{d_i\}_{i \in \mathbb{N}}$ in D

we want to prove $apply_f \left(\bigsqcup_i d_i \right) = \bigsqcup_i apply_f(d_i)$

$apply_f(\bigsqcup_i d_i) = apply(f, \bigsqcup_i d_i)$ by def of $apply_f$

$= f(\bigsqcup_i d_i)$ by def of $apply$

$= \bigsqcup_i f(d_i)$ by continuity of f

$= \bigsqcup_i apply(f, d_i)$ by def of $apply$

$= \bigsqcup_i apply_f(d_i)$ by def of $apply_f$

2. for any $d \in D$ $apply_d \triangleq \lambda f. f(d)$ is continuous

take $d \in D$ and a chain $\{f_i\}_{i \in \mathbb{N}}$ in $[D \rightarrow E]$

we want to prove $apply_d \left(\bigsqcup_i f_i \right) = \bigsqcup_i apply_d(f_i)$

$$apply_d(\bigsqcup_i f_i) = apply(\bigsqcup_i f_i, d) \quad \text{by def of } apply_d$$

$$= (\bigsqcup_i f_i)(d) \quad \text{by def of } apply$$

$$= \bigsqcup_i f_i(d) \quad \text{by def of lub of functions}$$

$$= \bigsqcup_i apply(f_i, d) \quad \text{by def of } apply$$

$$= \bigsqcup_i apply_d(f_i) \quad \text{by def of } apply_d$$

Apply: recap

$$\begin{array}{l} (D, \sqsubseteq_D) \\ (E, \sqsubseteq_E) \end{array} \quad \text{CPO} \quad \begin{array}{l} \textit{apply} : [D \rightarrow E] \times D \rightarrow E \\ \textit{apply}(f, d) \triangleq f(d) \end{array}$$

$$\textit{apply} \in [[D \rightarrow E] \times D \rightarrow E]$$

Fix

(D, \sqsubseteq_D) CPO $_{\perp}$

$fix : [D \rightarrow D] \rightarrow D$

$fix \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$

TH. fix is monotone

(try to prove on your own)

TH. fix is continuous

proof. $fix \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D) = \bigsqcup_{n \in \mathbb{N}} \lambda f. f^n(\perp_D)$

by def of lub in functional domains

we prove that $\forall n. \lambda f. f^n(\perp_D)$ is continuous

(by mathematical induction on n)

then fix is continuous because lub of continuous functions

(see next slides)

(continue) $\forall n. \lambda f. f^n(\perp_D)$

base case: $\lambda f. f^0(\perp_D) = \lambda f. \perp_D$

is a constant function (continuous)

inductive case: assume $g \triangleq \lambda f. f^n(\perp_D)$ is continuous

we want to prove $h \triangleq \lambda f. f^{n+1}(\perp_D)$ is continuous

take a chain $\{f_i\}_{i \in \mathbb{N}}$ in $[D \rightarrow D]$

we want to prove $h \left(\bigsqcup_{i \in \mathbb{N}} f_i \right) = \bigsqcup_{i \in \mathbb{N}} h(f_i)$

(see next slide)

(continue) $\forall n. \lambda f. f^n(\perp_D)$

$$g \triangleq \lambda f. f^n(\perp_D) \quad h \triangleq \lambda f. f^{n+1}(\perp_D) \quad h \left(\bigsqcup_{i \in \mathbb{N}} f_i \right) = \bigsqcup_{i \in \mathbb{N}} h(f_i)$$

$$\begin{aligned} h(\bigsqcup_i f_i) &= (\bigsqcup_i f_i)^{n+1}(\perp_D) && \text{by def of } h \\ &= (\bigsqcup_j f_j)((\bigsqcup_i f_i)^n(\perp_D)) && \text{by def of } (\cdot)^{n+1} \\ &= (\bigsqcup_j f_j)(g(\bigsqcup_i f_i)) && \text{by def of } g \\ &= (\bigsqcup_j f_j)(\bigsqcup_i g(f_i)) && \text{by ind. hyp (} g \text{ continuous)} \\ &= (\bigsqcup_j f_j)(\bigsqcup_i f_i^n(\perp_D)) && \text{by def of } g \\ &= \bigsqcup_j f_j(\bigsqcup_i f_i^n(\perp_D)) && \text{by def of lub in functional CPO} \\ &= \bigsqcup_j \bigsqcup_i f_j(f_i^n(\perp_D)) && \text{by continuity of } f_j \\ &= \bigsqcup_k f_k(f_k^n(\perp_D)) && \text{by switch lemma} \\ &= \bigsqcup_k f_k^{n+1}(\perp_D) && \text{by def of } (\cdot)^{n+1} \\ &= \bigsqcup_k h(f_k) && \text{by def of } h \end{aligned}$$

Fix: recap

(D, \sqsubseteq_D) CPO $_{\perp}$

$$\begin{aligned} \text{fix} &: [D \rightarrow D] \rightarrow D \\ \text{fix} &\triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D) \end{aligned}$$

$$\text{fix} \in [[D \rightarrow D] \rightarrow D]$$

Curry

(D, \sqsubseteq_D)

(E, \sqsubseteq_E) CPO

(F, \sqsubseteq_F)

$\text{curry} : (D \times E \rightarrow F) \rightarrow D \rightarrow E \rightarrow F$

$\text{curry } f \ d \ e \triangleq f(d, e)$

TH. f continuous \Rightarrow $\text{curry}(f)$ continuous

(try to prove on your own)

Uncurry

$$\begin{array}{l} (D, \sqsubseteq_D) \\ (E, \sqsubseteq_E) \text{ CPO} \\ (F, \sqsubseteq_F) \end{array} \quad \begin{array}{l} \text{uncurry} : (D \rightarrow E \rightarrow F) \rightarrow (D \times E) \rightarrow F \\ \text{uncurry } f (d, e) \triangleq f d e \end{array}$$

TH. f continuous \Rightarrow $\text{uncurry}(f)$ continuous

(try to prove on your own)

TH. uncurry is the inverse of curry

(try to prove on your own)

Disjoint Union



$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E) \quad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} + \mathcal{E} = (D \uplus E, \sqsubseteq_{D \uplus E})$$

$$D \uplus E \triangleq \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$$

how to order elements?

is there a bottom element?

is it a complete order?

how to define (continuous) injections?

$$\iota_D : D \rightarrow D \uplus E$$

$$\iota_E : E \rightarrow D \uplus E$$