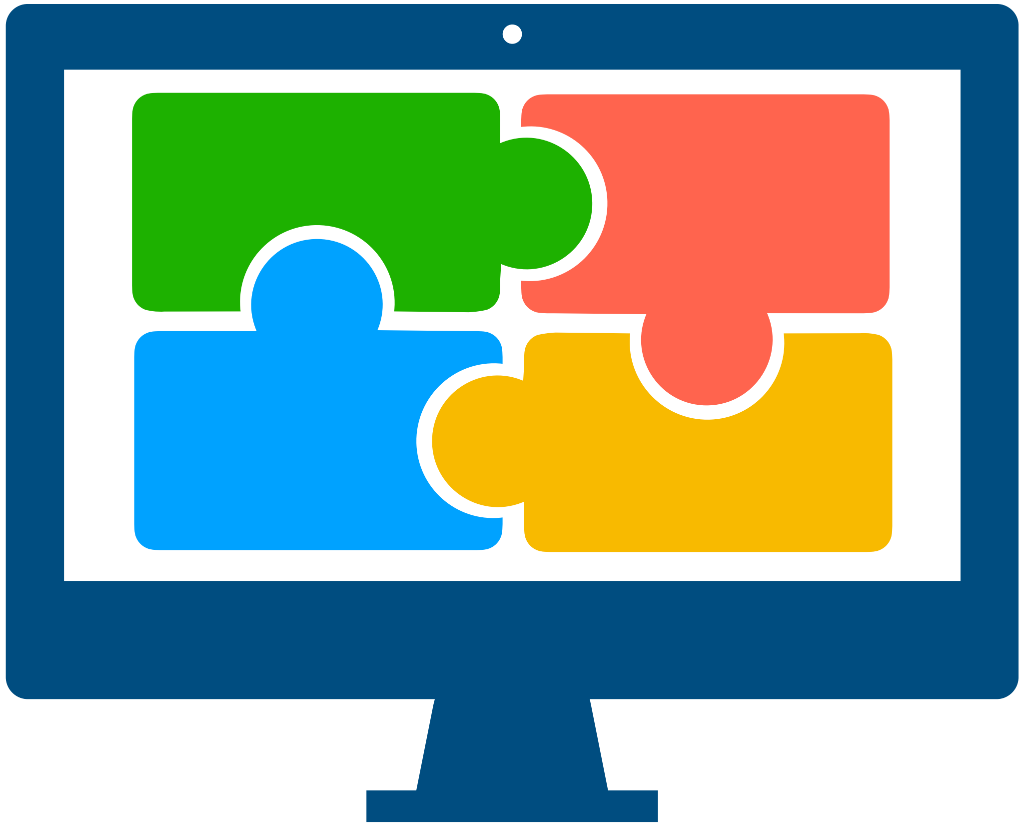

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**PSC 2020/21** (375AA, 9CFU)

Principles for Software Composition

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# 08c - Immediate Consequences Operator

# Recap

# Kleene's theorem

$$p \triangleq f(p)$$

a recursive definition

is there such a $p$?

is there a least $p$?

if $f : D \to D$ continuous $D$ $\mathrm{CPO}_\perp$

then $p \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$

also written $\mathit{fix}(f)$

# McCarthy 91 function

$f = M\ f$ $\quad(\mathbf{Pf}(\mathbb{N},\mathbb{N}),\sqsubseteq)$ $\quad\bot = \varnothing$ $\quad\mathrm{CPO}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

*M* monotone? $\quad f \sqsubseteq g \overset{?}{\Rightarrow} M(f) \sqsubseteq M(g)$

*M* continuous? $\quad M\left(\bigsqcup_i f_i\right) \overset{?}{\sqsubseteq} \bigsqcup_i M(f_i)$

# McCarthy 91 function

$$M\ f\ n = \begin{cases} n-10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

*M* monotone? ✓

we assume $f \sqsubseteq g$ and prove $M(f) \sqsubseteq M(g)$

take $(n, m) \in M(f)$ we prove $(n, m) \in M(g)$

if $n > 100$ $m = n - 10$ and $(n, n-10) \in M(g)$

if $n \leq 100$ $m = f(f(n+11))$

thus $\exists k.\ \{(n+11, k), (k, m)\} \in f \subseteq g$

and $(n, m) \in M(g)$

# McCarthy 91 function

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

*M* continuous? ✓

$$M\left(\bigsqcup_i f_i\right) \overset{?}{\sqsubseteq} \bigsqcup_i M(f_i)$$

let $f \triangleq \bigsqcup_i f_i$

take $(n, m) \in M(f)$

we prove $(n, m) \in \bigsqcup_i M(f_i)$

if $n > 100$ $\quad m = n - 10$ and $(n, n-10) \in M(f_1) \subseteq \bigsqcup_i M(f_i)$

if $n \leq 100$ $\quad m = f(f(n+11))$

thus $\exists k.\ \{(n+11, k), (k, m)\} \in f$

thus $\exists j_1, j_2.\ (n+11, k) \in f_{j_1}, (k, m) \in f_{j_2}$

let $j \triangleq \max\{j_1, j_2\}$ thus $\{(n+11, k), (k, m)\} \in f_j$

and $(n, m) \in M(f_j) \subseteq \bigsqcup_i M(f_i)$

# McCarthy 91 function

$f = M\ f$ $\quad(\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\quad\bot = \varnothing$ $\quad\mathrm{CPO}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

$$M^0(\varnothing) = \varnothing$$

$$M^1(\varnothing) = \{(n, n-10) \mid n > 100\} = \{(101, 91), (102, 92), ...\}$$

$$M^2(\varnothing) = \{(100, 91), (101, 91), (102, 92), ...\}$$

$$M^3(\varnothing) = \{(99, 91), (100, 91), (101, 91), (102, 92), ...\}$$

$$\cdots$$

$$M^{11}(\varnothing) = \{(91, 91), (92, 91), ..., (101, 91), (102, 92), ...\}$$

$$M^{12}(\varnothing) = \{(80, 91), (81, 91), ..., (101, 91), (102, 92), ...\}$$

$$\cdots$$

$$M^{20}(\varnothing) = \{(0, 91), (1, 91), ..., (101, 91), (102, 92), ...\}$$

fixpoint reached! (total function)

# Alternative perspective

$f = M\ f$ $\quad (\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\quad \bot = \varnothing$ $\quad \mathrm{CPO}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

formulas: $(n, m) \in f$

or just $(n, m)$ for brevity

$$R_M \triangleq \left\{ \frac{}{(n, n-10)}\ n > 100 \quad , \quad \frac{(n+11, k) \quad (k, m)}{(n, m)}\ n \leq 100 \right\}$$

$$f(n) = m \qquad \Leftrightarrow \qquad (n, m) \in I_{R_M}$$

(set of theorems of $R_M$)

# Immediate Consequences Operator (ICO)

# Immediate Consequences Operator

$F$ a set of formulas $(\wp(F), \subseteq)$ $\mathrm{CPO}_\perp$

$R$ a logical system $\widehat{R} : \wp(F) \to \wp(F)$ ICO

(with some restrictions)

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

$\widehat{R}$ applies the rules to the facts in $S$ in all possible ways

$\widehat{R}(S)$: all conclusions we can draw in one step
from hypotheses on $S$

We prove the least fixpoint of $\widehat{R}$ is $I_R$
(set of theorems of $R$)

# Example

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}}, \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \emptyset$$

$$\widehat{R}(S) = \{ \ \epsilon \in \mathcal{L} \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R. \ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

# Exercise

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \, \epsilon \in \mathcal{L} \, \}$$

$$\widehat{R}(S) = \text{?} \, \{ \, \epsilon \in \mathcal{L} \, , \; () \in \mathcal{L} \, \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \; ... \; x_n}{y} \in R. \; \{x_1, ..., x_n\} \subseteq S \right\}$$

# Exercise

Strings of balanced parentheses $F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \,,\, \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \,,\, \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \; () \in \mathcal{L} \; \}$$

$$\widehat{R}(S) = \text{?} \{ \; \epsilon \in \mathcal{L} \;,\; (()) \in \mathcal{L} \;,\; ()() \in \mathcal{L} \; \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \;\ldots\; x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

# Exercise

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \,,\; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \,,\; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{\ )( \in \mathcal{L}\ \}$$

$$\widehat{R}(S) = \text{?}\ \{\ \epsilon \in \mathcal{L}\ ,\ ()() \in \mathcal{L}\ ,\ )()( \in \mathcal{L}\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1\ \ldots\ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

# Example

Strings of balanced parentheses $F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \; , \; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \; , \; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \; )( \in \mathcal{L} \; , \; () \in \mathcal{L} \; \}$$

$$\widehat{R}(S) = \{ \; \epsilon \in \mathcal{L} \; , \; ()() \in \mathcal{L} \; , \; (()) \in \mathcal{L} \; , \; )()( \in \mathcal{L} \; , \; )(() \in \mathcal{L} \; , \; ())( \in \mathcal{L} \; \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \; \dots \; x_n}{y} \in R. \; \{x_1, \dots, x_n\} \subseteq S \right\}$$

# Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \frac{}{(0,0)} \ , \ \frac{}{(1,1)} \ , \ \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \right\}$$

$$S = \emptyset$$

$$\widehat{R}(S) = \{ \ (0,0) \ , \ (1,1) \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R. \ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

# Exercise

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\quad$ $F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \frac{}{(0,0)} \;,\; \frac{}{(1,1)} \;,\; \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \right\}$$

$$S = \{\ (2,1)\ \}$$

$$\widehat{R}(S) = \text{?}\,\{\ (0,0)\ ,\ (1,1)\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

# Exercise

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\qquad$ $F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \frac{}{(0,0)} \; , \; \frac{}{(1,1)} \; , \; \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \right\}$$

$$S = \{ \; (5,5) \; , \; (6,8) \; \}$$

$$\widehat{R}(S) = \; ? \, \{ \; (0,0) \; , \; (1,1) \; , \; (7,13) \; \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \; \ldots \; x_n}{y} \in R. \; \{x_1, \ldots, x_n\} \subseteq S \right\}$$

# ICO is monotone

**TH.** $\widehat{R}$ is monotone

proof. Take $S_1 \subseteq S_2$

we want to prove $\widehat{R}(S_1) \subseteq \widehat{R}(S_2)$

take $y \in \widehat{R}(S_1)$ we want to prove $y \in \widehat{R}(S_2)$

$$\Downarrow$$

$\exists \dfrac{x_1 \ \dots \ x_n}{y} \in R$ with $\{x_1, \dots, x_n\} \subseteq S_1$

$$\Downarrow \quad S_1 \subseteq S_2$$

$$\{x_1, \dots, x_n\} \subseteq S_2$$

$$\Downarrow$$

$$y \in \widehat{R}(S_2)$$

# ICO is continuous

**TH.** $\widehat{R}$ is continuous (under some hypotheses)

proof. Take $\{S_i\}_{i\in\mathbb{N}}$ a chain in $\wp(F)$

We want to prove $\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) = \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$

$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \subseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$ because $\widehat{R}$ is monotone

$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \supseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$ ?

# ICO is continuous (ctd)

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \supseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$

Take $y \in \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$ we want to prove $y \in \bigcup_{i\in\mathbb{N}} \widehat{R}(S_i)$

$$\Downarrow$$

$\exists \frac{x_1 \; \ldots \; x_n}{y} \in R$ with $\{x_1, \ldots, x_n\} \subseteq \bigcup_{i\in\mathbb{N}} S_i$

thus $\forall j \in [1, n].\ \exists k_j \in \mathbb{N}.\ x_j \in S_{k_j}$ take $k = \max\{k_1, \ldots, k_n\}$

clearly $\{x_1, \ldots, x_n\} \subseteq S_k$

thus $y \in \widehat{R}(S_k) \subseteq \bigcup_{i\in\mathbb{N}} \widehat{R}(S_i)$

**possible iff each rule has finitely many premises**

# Provable theorems

$I_R$ set of theorems in $R$

$I_R^n$ theorems provable with a derivation of height at most $n$

$$I_R^0 = \emptyset$$

$$I_R^{n+1} = I_R^n \cup \widehat{R}(I_R^n)$$

provable using one more inference step

theorems provable with a derivation of height at most $n$

clearly $I_R = \bigcup_{n \in \mathbb{N}} I_R^n$

# n-depth theorems

**TH.** Let $P(n) \triangleq I_R^n = \widehat{R}^n(\emptyset)$ $\forall n \in \mathbb{N}.\ P(n)$

proof. by mathematical induction

$P(0) \triangleq I_R^0 = \widehat{R}^0(\emptyset)$ $I_R^0 = \emptyset = \widehat{R}^0(\emptyset)$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

take a generic $n$
we assume $P(n) \triangleq I_R^n = \widehat{R}^n(\emptyset)$

we want to prove $P(n+1) \triangleq I_R^{n+1} = \widehat{R}^{n+1}(\emptyset)$

$$\begin{aligned}
I_R^{n+1} &= I_R^n \cup \widehat{R}(I_R^n) && \text{by def} \\
&= \widehat{R}^n(\emptyset) \cup \widehat{R}(\widehat{R}^n(\emptyset)) && \text{by inductive hypothesis} \\
&= \widehat{R}^n(\emptyset) \cup \widehat{R}^{n+1}(\emptyset) && \text{by def} \\
&= \widehat{R}^{n+1}(\emptyset) && \text{because } \widehat{R}^n(\emptyset) \subseteq \widehat{R}^{n+1}(\emptyset)
\end{aligned}$$

# ICO's fixpoint

**TH.** $\mathit{fix}(\widehat{R}) = I_R$ (under some hypotheses)

**each rule must have finitely many premises**

proof.

by Kleene's fixpoint theorem we know that $\mathit{fix}(\widehat{R})$ exists

$$\mathit{fix}(\widehat{R}) = \bigcup_{n\in\mathbb{N}} \widehat{R}^n(\emptyset)$$ by def

$$= \bigcup_{n\in\mathbb{N}} I_R^n$$ by previous result

$$= I_R$$ by def

# Example

Strings of balanced parentheses $F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

$$R = \left\{ \ \frac{}{\epsilon \in \mathcal{L}} \ , \ \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \ , \ \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \ \right\}$$

$$\widehat{R}^0(\emptyset) = \emptyset$$

$$\widehat{R}^1(\emptyset) = \{ \ \epsilon \in \mathcal{L} \ \}$$

$$\widehat{R}^2(\emptyset) = \{ \ \epsilon \in \mathcal{L} \ , \ () \in \mathcal{L} \ \}$$

$$\widehat{R}^3(\emptyset) = \{ \ \epsilon \in \mathcal{L} \ , \ () \in \mathcal{L} \ , \ (()) \in \mathcal{L} \ , \ ()() \in \mathcal{L} \ \}$$

$$\begin{aligned} \widehat{R}^4(\emptyset) = \{ \ & \epsilon \in \mathcal{L} \ , \ () \in \mathcal{L} \ , \ (()) \in \mathcal{L} \ , \ ()() \in \mathcal{L} \ , \ ((())) \in \mathcal{L} \ , \\ & (()()) \in \mathcal{L} \ , \ ()(()) \in \mathcal{L} \ , \ (())() \in \mathcal{L} \ , \ ()()() \in \mathcal{L} \ , \\ & (())()() \in \mathcal{L} \ , \ ()()(()) \in \mathcal{L} \ , \ (())(()) \in \mathcal{L} \ , \ ()()()() \in \mathcal{L} \ \} \end{aligned}$$

# Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\qquad$ $F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)} \ , \ \frac{}{(1,1)} \ , \ \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \ \right\}$$

$$\widehat{R}^0(\emptyset) = \emptyset$$

$$\widehat{R}^1(\emptyset) = \{ \ (0,0) \ , \ (1,1) \ \}$$

$$\widehat{R}^2(\emptyset) = \{ \ (0,0) \ , \ (1,1) \ , \ (2,1) \ \}$$

$$\widehat{R}^3(\emptyset) = \{ \ (0,0) \ , \ (1,1) \ , \ (2,1) \ , \ (3,2) \ \}$$

$$\widehat{R}^4(\emptyset) = \{ \ (0,0) \ , \ (1,1) \ , \ (2,1) \ , \ (3,2) \ , \ (4,3) \ \}$$

$$\widehat{R}^5(\emptyset) = \{ \ (0,0) \ , \ (1,1) \ , \ (2,1) \ , \ (3,2) \ , \ (4,3) \ , \ (5,5) \ \}$$

$$\widehat{R}^6(\emptyset) = \{ \ (0,0) \ , \ (1,1) \ , \ (2,1) \ , \ (3,2) \ , \ (4,3) \ , \ (5,5) \ , \ (6,8) \ \}$$