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**PSC 2020/21** (375AA, 9CFU)

Principles for Software Composition

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**05b - More Induction**

# Determinacy by structural induction

# Determinacy of arithmetic expressions

$a ::= x \mid n \mid a \text{ op } a$

$x \in \text{Ide} \quad \text{op} \in \{+, \times, -\}$

$n \in \mathbb{Z} \quad \mathbb{M} \triangleq \{\sigma \mid \sigma : \text{Ide} \rightarrow \mathbb{Z}\}$

$$\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)} \quad \frac{}{\langle n, \sigma \rangle \longrightarrow n} \quad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \text{ op } n_1}$$

$$P(a) \triangleq \forall \sigma \in \mathbb{M}. \forall m, m' \in \mathbb{Z}. \langle a, \sigma \rangle \longrightarrow m \wedge \langle a, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

$\forall a. P(a) ?$

# Structural induction principle

$$\forall x \in \text{Ide. } P(x)$$

$$\forall n \in \mathbb{Z}. P(n)$$

$$\forall a_0, a_1. P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$$

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$$\forall a. P(a)$$

# Base case

$\forall x \in \text{Ide. } P(x)$

Take a generic  $x \in \text{Ide}$

We want to prove

$$P(x) \stackrel{\Delta}{=} \forall \sigma, m, m'. \langle x, \sigma \rangle \longrightarrow m \wedge \langle x, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic  $\sigma, m, m'$  s.t.  $\langle x, \sigma \rangle \longrightarrow m$  and  $\langle x, \sigma \rangle \longrightarrow m'$

We want to prove  $m = m'$

Consider the goal  $\langle x, \sigma \rangle \longrightarrow m$

Only the rule  $\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$  is applicable, hence  $m = \sigma(x)$

Similarly, since  $\langle x, \sigma \rangle \longrightarrow m'$  it must be  $m' = \sigma(x)$

and thus we conclude  $m = m'$

# Base case

$\forall n \in \mathbb{Z}. P(n)$

Take a generic  $n \in \mathbb{Z}$

We want to prove

$$P(n) \triangleq \forall \sigma, m, m'. \langle n, \sigma \rangle \longrightarrow m \wedge \langle n, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic  $\sigma, m, m'$  s.t.  $\langle n, \sigma \rangle \longrightarrow m$  and  $\langle n, \sigma \rangle \longrightarrow m'$

We want to prove  $m = m'$

Consider the goal  $\langle n, \sigma \rangle \longrightarrow m$

Only the rule  $\frac{}{\langle n, \sigma \rangle \longrightarrow n}$  is applicable, hence  $m = n$

Similarly, since  $\langle n, \sigma \rangle \longrightarrow m'$  it must be  $m' = n$

and thus we conclude  $m = m'$

# Inductive case

$\forall a_0, a_1. P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$  Take generic  $a_0, a_1$

We assume (inductive hypotheses)

$$P(a_i) \triangleq \forall \sigma, m_i, m'_i. \langle a_i, \sigma \rangle \longrightarrow m_i \wedge \langle a_i, \sigma \rangle \longrightarrow m'_i \Rightarrow m_i = m'_i$$

We want to prove

$$P(a_0 \text{ op } a_1) \triangleq \forall \sigma, m, m'. \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m \wedge \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic  $\sigma, m, m'$  such that  $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$  and  $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m'$

We want to prove  $m = m'$

# Inductive case (ctd)

Consider the goal  $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$

Only the rule  $\frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \text{ op } n_1}$  is applicable

hence  $m = n_0 \text{ op } n_1$  with  $\langle a_0, \sigma \rangle \longrightarrow n_0$  and  $\langle a_1, \sigma \rangle \longrightarrow n_1$

Similarly, since  $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m'$

it must be  $m' = n'_0 \text{ op } n'_1$  with  $\langle a_0, \sigma \rangle \longrightarrow n'_0$  and  $\langle a_1, \sigma \rangle \longrightarrow n'_1$

By inductive hypotheses,  $n_0 = n'_0$  and  $n_1 = n'_1$

and thus we conclude  $m = n_0 \text{ op } n_1 = n'_0 \text{ op } n'_1 = m'$



**Many sorted signatures**

# Terms over a signature

$S = \{s, \dots\}$  a set of **sorts**

$\Sigma = \{\Sigma_{s_1 \dots s_n, s}\}_{s_1, \dots, s_n, s \in S}$  a **many sorted signature**

$f \in \Sigma_{s_1 \dots s_n, s}$        $f : (s_1 \times \dots \times s_n) \rightarrow s$

$T_{\Sigma, s}$  denotes the set of all **terms of sort s**

it is the least set such that:

- if  $c \in \Sigma_{\epsilon, s}$ , then  $c \in T_{\Sigma, s}$
- if  $f \in \Sigma_{s_1 \dots s_n, s}$  and  $\forall i. t_i \in T_{\Sigma, s_i}$ , then  $f(t_1, \dots, t_n) \in T_{\Sigma, s}$

$T_{\Sigma} = \{T_{\Sigma, s}\}_{s \in S}$  denotes the set of all **well-sorted terms**

# Boolean expressions

$$x \in \text{Ide} \quad n \in \mathbb{Z} \quad \text{op} \in \{+, \times, -\}$$

$$v \in \mathbb{B} \quad \text{bop} \in \{\wedge, \vee\} \quad \text{cmp} \in \{<, \leq, >, \geq, =, \neq\}$$

$$a ::= x \mid n \mid a \text{ op } a$$

$$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$$

$$S \triangleq \{\text{Aexp}, \text{Bexp}\}$$

$$\Sigma_{\epsilon, \text{Aexp}} \triangleq \text{Ide} \cup \mathbb{Z}$$

$$\Sigma_{\text{Aexp} \text{Aexp}, \text{Aexp}} \triangleq \{+, \times, -\}$$

$$\Sigma_{\epsilon, \text{Bexp}} \triangleq \mathbb{B}$$

$$\Sigma_{\text{Aexp} \text{Aexp}, \text{Bexp}} \triangleq \{<, \leq, >, \geq, =, \neq\}$$

$$\Sigma_{\text{Bexp}, \text{Bexp}} \triangleq \{\neg\}$$

$$\Sigma_{\text{Bexp} \text{Bexp}, \text{Bexp}} \triangleq \{\wedge, \vee\}$$

# Semantics of expressions

$a ::= x \mid n \mid a \text{ op } a$

$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$

$$\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)} \quad \frac{}{\langle n, \sigma \rangle \longrightarrow n} \quad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \text{ op } n_1}$$

$$\frac{}{\langle v, \sigma \rangle \longrightarrow v} \quad \frac{\langle b, \sigma \rangle \longrightarrow v}{\langle \neg b, \sigma \rangle \longrightarrow \neg v} \quad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow n_0 \text{ cmp } n_1}$$

$$\frac{\langle b_0, \sigma \rangle \longrightarrow v_0 \quad \langle b_1, \sigma \rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1, \sigma \rangle \longrightarrow v_0 \text{ bop } v_1}$$

# Subterms

$$t_i \prec f(t_1, \dots, t_n)$$

sometimes, an additional requirement:  
sort-preserving relation

$$f \in \Sigma_{s_1 \dots s_n, s} \wedge s_i = s$$

# One sorted = Unsorted

a special case:

$S = \{*\}$  a singleton set of **sorts**

$$\Sigma_{\underbrace{*\dots*}_n,*} \simeq \Sigma_n$$

# Termination of boolean expressions

# Termination of expressions

$a ::= x \mid n \mid a \text{ op } a$

$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$

$$\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)} \quad \frac{}{\langle n, \sigma \rangle \longrightarrow n} \quad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \text{ op } n_1}$$

$$\frac{}{\langle v, \sigma \rangle \longrightarrow v} \quad \frac{\langle b, \sigma \rangle \longrightarrow v}{\langle \neg b, \sigma \rangle \longrightarrow \neg v} \quad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow n_0 \text{ cmp } n_1}$$

$$\frac{\langle b_0, \sigma \rangle \longrightarrow v_0 \quad \langle b_1, \sigma \rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1, \sigma \rangle \longrightarrow v_0 \text{ bop } v_1}$$

$$P(b) \triangleq \forall \sigma \in \mathbb{M}. \exists v \in \mathbb{B}. \langle b, \sigma \rangle \longrightarrow v$$

$$\forall b. P(b) ?$$



# Base case

$\forall v \in \mathbb{B}. P(v)$

Take a generic  $v \in \mathbb{B}$

We want to prove  $P(v) \triangleq \forall \sigma. \exists u. \langle v, \sigma \rangle \longrightarrow u$

Take a generic  $\sigma \in \mathbb{M}$  and consider the goal  $\langle v, \sigma \rangle \longrightarrow u$

the only  
variable

By rule  $\frac{}{\langle v, \sigma \rangle \longrightarrow v}$

we have  $\langle v, \sigma \rangle \longrightarrow u \xleftarrow{[u=v]} \square$

And we are done (taking  $u = v$ )

# A surprising base case

$\forall a_0, a_1. P(a_0 \text{ cmp } a_1)$

Take generic  $a_0, a_1$

We want to prove  $P(a_0 \text{ cmp } a_1) \triangleq \forall \sigma. \exists v. \langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow v$

Consider the goal  $\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow v$

By rule  $\frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow n_0 \text{ cmp } n_1}$  we have

$\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow v \leftarrow_{[v=n_0 \text{ cmp } n_1]} \langle a_0, \sigma \rangle \longrightarrow n_0, \langle a_1, \sigma \rangle \longrightarrow n_1$

By **termination of arithmetic expressions**, such  $n_0, n_1$  exist

And we are done (taking  $v = n_0 \text{ cmp } n_1$ )

# Proof obligations



try to complete on your own  
the inductive cases of the proof

$$\forall b. P(b) \Rightarrow P(\neg b)$$

$$\forall b_0, b_1. (P(b_0) \wedge P(b_1)) \Rightarrow P(b_0 \text{ bop } b_1)$$

# Commands

# Commands

$a ::= x \mid n \mid a + a \mid \dots$

$b ::= v \mid a \leq a \mid \dots$

$c ::= \mathbf{skip} \mid x := a \mid c; c \mid \mathbf{if } b \mathbf{ then } c \mathbf{ else } c \mid \mathbf{while } b \mathbf{ do } c$

$S \triangleq \{ \text{Aexp}, \text{Bexp}, \text{Com} \}$

$\Sigma_{\epsilon, \text{Aexp}} \triangleq \text{Ide} \cup \mathbb{Z}$

$\Sigma_{\text{AexpAexp}, \text{Aexp}} \triangleq \{ +, \times, - \}$

$\Sigma_{\epsilon, \text{Bexp}} \triangleq \mathbb{B}$

$\Sigma_{\text{AexpAexp}, \text{Bexp}} \triangleq \{ <, \leq, >, \geq, =, \neq \}$

$\Sigma_{\text{Bexp}, \text{Bexp}} \triangleq \{ \neg \}$

$\Sigma_{\text{BexpBexp}, \text{Bexp}} \triangleq \{ \wedge, \vee \}$

$\Sigma_{\epsilon, \text{Com}} \triangleq \{ \mathbf{skip} \}$

$\Sigma_{\text{Aexp}, \text{Com}} \triangleq \{ x := \mid x \in \text{Ide} \}$

$\Sigma_{\text{ComCom}, \text{Com}} \triangleq \{ ; \}$

$\Sigma_{\text{BexpComCom}, \text{Com}} \triangleq \{ \mathbf{if} \}$

$\Sigma_{\text{BexpCom}, \text{Com}} \triangleq \{ \mathbf{while} \}$

# Memories

$$\mathbb{M} \triangleq \{ \sigma : \text{Ide} \rightarrow \mathbb{Z} \mid \sigma \text{ has finite support} \}$$

$$\{x \in \text{Ide} \mid \sigma(x) \neq 0\} \text{ is finite}$$

$$(n_1/x_1, \dots, n_k/x_k) : \text{Ide} \rightarrow \mathbb{Z}$$

all different

$$(n_1/x_1, \dots, n_k/x_k)(x) \triangleq \begin{cases} n_i & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

$\sigma_0 \triangleq ()$  is the typical initial memory

# Memory updates

$$\sigma[n/y](x) \triangleq \begin{cases} n & \text{if } x = y \\ \sigma(x) & \text{otherwise} \end{cases}$$

$$\forall \sigma, m, n, y. \sigma[m/y][n/y] = \sigma[n/y]$$

$$\sigma[m/y][n/y](x) \triangleq \begin{cases} n & \text{if } x = y \\ \sigma[m/y](x) = \sigma(x) & \text{otherwise} \end{cases}$$

$$\forall \sigma, m, n, y, z. y \neq z \Rightarrow \sigma[n/y][m/z] = \sigma[m/z][n/y]$$

we write  $\sigma[n/y, m/z]$  in such cases

$$(n_1/x_1, \dots, n_k/x_k) = \sigma_0[n_1/x_1, \dots, n_k/x_k]$$

# Semantics of commands

$c ::= \text{skip} \mid x := a \mid c; c \mid \text{if } b \text{ then } c \text{ else } c \mid \text{while } b \text{ do } c$

$$\frac{}{\langle \text{skip}, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]} \quad \frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0; c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff} \quad \langle c_1, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c_0, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \text{while } b \text{ do } c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma'}$$



# Recursive definition!

one premise is as much complex as the conclusion



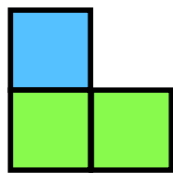
$$\frac{\langle b, \sigma \rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \mathbf{while } b \mathbf{ do } c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \mathbf{while } b \mathbf{ do } c, \sigma \rangle \longrightarrow \sigma'}$$

# Triangular numbers

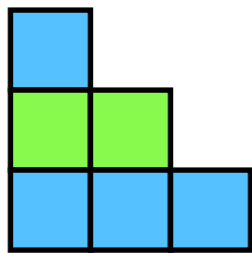
$T_1$  1



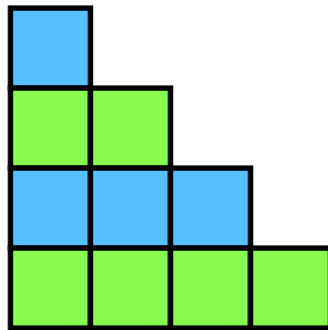
$T_2$  3



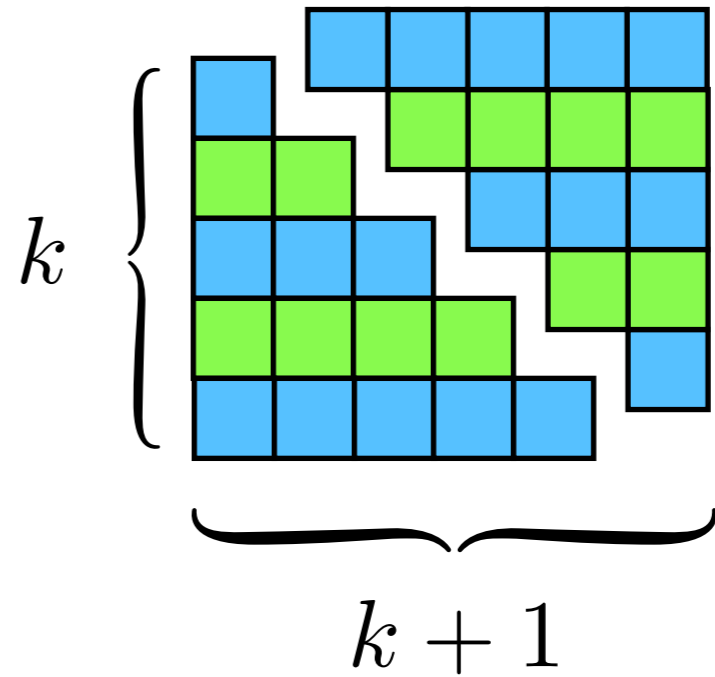
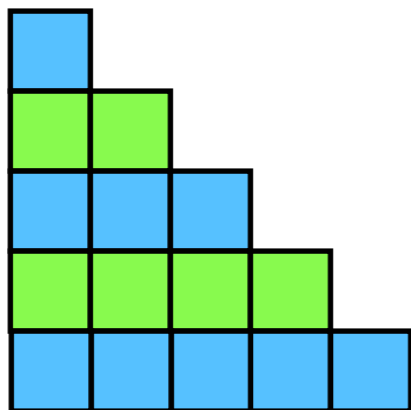
$T_3$  6



$T_4$  10



$T_5$  15



$$T_k \triangleq \sum_{i=1}^k i = \frac{k(k+1)}{2}$$

# Compute $T_2$ without div

$$\begin{array}{l} c_0 \{ \\ c_1 \{ \\ w \{ \end{array} \left\{ \begin{array}{l} t := 0; \\ i := 1; \overbrace{b} \\ \mathbf{while} \ i \leq k \ \mathbf{do} \ ( \\ \quad t := t + i; \\ \quad i := i + 1 ) \end{array} \right. \left. \begin{array}{l} \} c_2 \\ \} c_3 \end{array} \right\} c$$

let's find  $\sigma$  such that  $\langle (c_0; c_1); w, (2/k) \rangle \longrightarrow \sigma$

let's find  $\sigma', \sigma$  such that  $\left\{ \begin{array}{l} \langle c_0; c_1, (2/k) \rangle \longrightarrow \sigma' \\ \langle w, \sigma' \rangle \longrightarrow \sigma \end{array} \right.$

# Let's derive

let's find  $\sigma'$  such that  $\langle c_0; c_1, (2/k) \rangle \longrightarrow \sigma'$

$\langle c_0; c_1, (2/k) \rangle \longrightarrow \sigma'$

$\swarrow \langle t := 0, (2/k) \rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1 \rangle \longrightarrow \sigma'$

$\swarrow \sigma_1 = (2/k)[n_1/t] \langle 0, (2/k) \rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t) \rangle \longrightarrow \sigma'$

$\swarrow n_1 = 0 \langle i := 1, (2/k, 0/t) \rangle \longrightarrow \sigma'$

$\swarrow \sigma' = (2/k, 0/t)[n_2/i] \langle 1, (2/k, 0/t) \rangle \longrightarrow n_2$

$\swarrow n_2 = 1 \square$

$\sigma' = (2/k, 0/t, 1/i)$

more concisely:  $\langle c_0; c_1, (2/k) \rangle \longrightarrow \sigma'$

$\swarrow^* \sigma' = (2/k, 0/t, 1/i) \square$

$c_0 \{ t := 0;$   
 $c_1 \{ i := 1; \overbrace{b} \text{ while } i \leq k \text{ do } ($   
 $w \{ \quad t := t + i; \quad \} c_2 \}$   
 $\quad i := i + 1) \quad \} c_3 \} c$   
 $\sigma \triangleq (2/n)$

# Let's derive

let's find  $\sigma$  such that  $\langle w, (2/k, 0/t, 1/i) \rangle \longrightarrow \sigma$

$\langle w, (2/k, 0/t, 1/i) \rangle \longrightarrow \sigma$

$\swarrow \langle b, (2/k, 0/t, 1/i) \rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i) \rangle \longrightarrow \sigma_1, \langle w, \sigma_1 \rangle \longrightarrow \sigma$

$\swarrow^* \langle c, (2/k, 0/t, 1/i) \rangle \longrightarrow \sigma_1, \langle w, \sigma_1 \rangle \longrightarrow \sigma$

$\swarrow_{\sigma_1=(2/k, 0/t, 1/i)[1/t, 2/i]}^* \langle w, (2/k, 1/t, 2/i) \rangle \longrightarrow \sigma$

$\swarrow \langle b, (2/k, 1/t, 2/i) \rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i) \rangle \longrightarrow \sigma_2, \langle w, \sigma_2 \rangle \longrightarrow \sigma$

$\swarrow^* \langle c, (2/k, 1/t, 2/i) \rangle \longrightarrow \sigma_2, \langle w, \sigma_2 \rangle \longrightarrow \sigma$

$\swarrow_{\sigma_2=(2/k, 1/t, 2/i)[3/t, 3/i]}^* \langle w, (2/k, 3/t, 3/i) \rangle \longrightarrow \sigma$

$\swarrow_{\sigma=(2/k, 3/t, 3/i)} \langle b, (2/k, 3/t, 3/i) \rangle \longrightarrow \mathbf{ff}$

$\swarrow^* \square$

$c_0 \{ t := 0;$   
 $c_1 \{ i := 1; \overbrace{b}^{\text{while } i \leq k \text{ do } (}$   
 $w \{ \begin{array}{l} t := t + i; \\ i := i + 1 \end{array} ) \} c_2 \} c_3 \} c$

$\sigma \triangleq (2/n)$

$\sigma = (2/k, 3/t, 3/i)$

# Divergence

# Termination of commands?

$c ::= \text{skip} \mid x := a \mid c; c \mid \text{if } b \text{ then } c \text{ else } c \mid \text{while } b \text{ do } c$

$$\frac{}{\langle \text{skip}, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]} \quad \frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0; c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff} \quad \langle c_1, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c_0, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \text{while } b \text{ do } c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma'}$$

$$P(c) \triangleq \forall \sigma \in \mathbb{M}. \exists \sigma' \in \mathbb{M}. \langle c, \sigma \rangle \longrightarrow \sigma'$$

$$\forall c. P(c) ?$$

# Termination?

$$P(c) \triangleq \forall \sigma. \exists \sigma'. \langle c, \sigma \rangle \longrightarrow \sigma' \qquad \forall c. P(c) ?$$

take  $w \triangleq$  while tt do skip

$$\langle w, \sigma \rangle \longrightarrow \sigma'$$

$$\swarrow \langle \mathbf{tt}, \sigma \rangle \longrightarrow \mathbf{tt}, \langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma'', \langle w, \sigma'' \rangle \longrightarrow \sigma'$$

$$\swarrow \langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma'', \langle w, \sigma'' \rangle \longrightarrow \sigma'$$

$$\swarrow_{\sigma'' = \sigma} \langle w, \sigma \rangle \longrightarrow \sigma'$$

same goal from which we started!  
not much options:  $\langle \mathbf{tt}, \sigma \rangle \not\rightarrow \mathbf{ff}$   
no way to complete the derivation!

$$\neg P(w)$$

we have found a counterexample to termination



# Divergence

$\langle c, \sigma \rangle \not\rightarrow$  means  $\neg \exists \sigma'. \langle c, \sigma \rangle \rightarrow \sigma'$

and we say that  $c$  diverges on  $\sigma$

sometimes divergence is very difficult to detect  
(well... it is undecidable)

# Divergence

take  $w \triangleq$  **while**  $x > 0$  **do**  $x := x + 1$

take a generic  $\sigma$

if  $\sigma(x) \leq 0$ :  $\langle w, \sigma \rangle \longrightarrow \sigma'$   $\nwarrow_{\sigma'=\sigma} \langle x > 0, \sigma \rangle \longrightarrow \mathbf{ff}$   $\nwarrow^* \square$

hence  $\langle w, \sigma \rangle \longrightarrow \sigma$

if  $\sigma(x) > 0$ :  $\langle w, \sigma \rangle \longrightarrow \sigma'$

$\nwarrow \langle x > 0, \sigma \rangle \longrightarrow \mathbf{tt}, \langle x := x + 1, \sigma \rangle \longrightarrow \sigma'', \langle w, \sigma'' \rangle \longrightarrow \sigma'$

$\nwarrow^* \langle x := x + 1, \sigma \rangle \longrightarrow \sigma'', \langle w, \sigma'' \rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma''=\sigma[n/x]} \langle x + 1, \sigma \rangle \longrightarrow n, \langle w, \sigma[n/x] \rangle \longrightarrow \sigma'$

$\nwarrow_{n=\sigma(x)+1}^* \langle w, \sigma[\sigma(x) + 1/x] \rangle \longrightarrow \sigma'$

not the same goal from which we started!  $\sigma(x) > 0 \Rightarrow \sigma[\sigma(x) + 1/x](x) = \sigma(x) + 1 > 0$   
 how can we prove divergence?

# Proving divergence (if possible)

consider  $w \triangleq \text{while } b \text{ do } c$

suppose we find a set of memories  $S \subseteq \mathbb{M}$  such that

- $\forall \sigma \in S. \langle b, \sigma \rangle \longrightarrow \text{tt}$
- $\forall \sigma \in S. \forall \sigma' \in \mathbb{M}. \langle c, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$

then we can conclude  $\forall \sigma \in S. \langle w, \sigma \rangle \not\rightarrow$

note that if  $\langle c, \sigma \rangle \not\rightarrow$ , then  $\langle c, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$  trivially holds

# Revisiting the example

take  $w \triangleq$  **while**  $x > 0$  **do**  $x := x + 1$

take a generic  $\sigma$

if  $\sigma(x) \leq 0$ :  $\langle w, \sigma \rangle \longrightarrow \sigma$

let  $S \triangleq \{\sigma \mid \sigma(x) > 0\}$

•  $\forall \sigma \in S. \langle x > 0, \sigma \rangle \longrightarrow \mathbf{tt}$  ✓

•  $\forall \sigma \in S. \forall \sigma'. \langle x := x + 1, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$  ✓

in fact  $\langle x := x + 1, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' = \sigma[\sigma(x) + 1/x]$

$$\sigma(x) > 0 \Rightarrow \sigma[\sigma(x) + 1/x](x) = \sigma(x) + 1 > 0$$

therefore, if  $\sigma(x) > 0$ , then  $\langle w, \sigma \rangle \not\rightarrow$



# Exercise

take  $w \triangleq$  **while**  $x \neq 0$  **do**  $x := x - 2$

find the set of all and only memories  $\sigma$  such that  $\langle w, \sigma \rangle \not\rightarrow$


$$S_1 \triangleq \{\sigma \mid \sigma(x) < 0\}$$

$$S_2 \triangleq \{\sigma \mid \exists k \in \mathbb{Z}. \sigma(x) = 2k + 1\}$$

- $\forall \sigma \in S. \langle x \neq 0, \sigma \rangle \longrightarrow \mathbf{tt}$
- $\forall \sigma \in S. \forall \sigma'. \langle x := x - 2, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$   
 $\langle x := x - 2, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' = \sigma[\sigma(x) - 2/x]$

# Collatz's conjecture: half or triple plus one

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 $w \triangleq$  while  $x > 1$  do ( if  $x \% 2 = 0$  then  $x := x/2$   
else  $x := (3 \times x) + 1$  )
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$\forall \sigma. \sigma(x) \leq 1 \Rightarrow \langle w, \sigma \rangle \longrightarrow \sigma$  

open conjecture:  $\forall \sigma. \exists \sigma'. \langle w, \sigma \rangle \longrightarrow \sigma'$   $\equiv \neg(\exists \sigma. \langle w, \sigma \rangle \not\rightarrow)$

more precisely:  $\forall \sigma. \sigma(x) > 1 \Rightarrow \langle w, \sigma \rangle \longrightarrow \sigma[1/x]$

## Experimental evidence:

as of 2020, the conjecture has been verified for all values up to  $2^{68}$ .

Barina, David. "Convergence verification of the Collatz problem".

The Journal of Supercomputing (2020). doi:10.1007/s11227-020-03368-x

Determinacy?

# Determinacy of commands?

$c ::= \text{skip} \mid x := a \mid c; c \mid \text{if } b \text{ then } c \text{ else } c \mid \text{while } b \text{ do } c$

$$\frac{}{\langle \text{skip}, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]} \quad \frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0; c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff} \quad \langle c_1, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c_0, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \text{while } b \text{ do } c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma'}$$

$$P(c) \stackrel{\Delta}{=} \forall \sigma, \sigma_1, \sigma_2. \langle c, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2 \quad \forall c. P(c) ?$$



# Structural induction principle

$$\forall x, a. P(x := a) \quad P(\mathbf{skip})$$

$$\forall c_0, c_1. P(c_0) \wedge P(c_1) \Rightarrow P(c_0 ; c_1)$$

$$\forall b, c_0, c_1. P(c_0) \wedge P(c_1) \Rightarrow P(\mathbf{if } b \mathbf{ then } c_0 \mathbf{ else } c_1)$$

$$\forall b, c. P(c) \Rightarrow P(\mathbf{while } b \mathbf{ do } c)$$

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$$\forall c \in \mathbf{Com}. P(c)$$

# Base case

$\forall x, a. P(x := a)$

Take generic  $x \in \text{Ide}, a \in \text{Aexp}$

We want to prove

$$P(x := a) \stackrel{\Delta}{=} \forall \sigma, \sigma_1, \sigma_2. \langle x := a, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle x := a, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Take generic  $\sigma, \sigma_1, \sigma_2$  s.t.  $\langle x := a, \sigma \rangle \longrightarrow \sigma_1$  and  $\langle x := a, \sigma \rangle \longrightarrow \sigma_2$

We want to prove  $\sigma_1 = \sigma_2$

Consider the goal  $\langle x := a, \sigma \rangle \longrightarrow \sigma_1$

Only the rule  $\frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]}$  is applicable, hence  $\sigma_1 = \sigma[n/x]$   
with  $\langle a, \sigma \rangle \longrightarrow n$

Similarly, since  $\langle x := a, \sigma \rangle \longrightarrow \sigma_2$  it must be  $\sigma_2 = \sigma[m/x]$   
with  $\langle a, \sigma \rangle \longrightarrow m$

by determinacy of Aexp we have  $n = m$  and thus  $\sigma_1 = \sigma_2$

# Inductive case

$$\forall c_0, c_1. P(c_0) \wedge P(c_1) \Rightarrow P(c_0 ; c_1)$$

Take generic  $c_0, c_1$

We assume

(inductive hypotheses)

$$P(c_i) \triangleq \forall \sigma, \sigma_1, \sigma_2. \langle c_i, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle c_i, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

We want to prove

$$P(c_0 ; c_1) \triangleq \forall \sigma, \sigma_1, \sigma_2. \langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Take generic  $\sigma, \sigma_1, \sigma_2$  such that  $\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_1$  and  $\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_2$

We want to prove  $\sigma_1 = \sigma_2$

# Inductive case (ctd)

Consider the goal  $\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_1$

Only the rule  $\frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma'}$  is applicable

hence  $\sigma_1 = \sigma'_1$  with  $\langle c_0, \sigma \rangle \longrightarrow \sigma''_1$  and  $\langle c_1, \sigma''_1 \rangle \longrightarrow \sigma'_1$

Similarly, since  $\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_2$

it must be  $\sigma_2 = \sigma'_2$  with  $\langle c_0, \sigma \rangle \longrightarrow \sigma''_2$  and  $\langle c_1, \sigma''_2 \rangle \longrightarrow \sigma'_2$

By inductive hypothesis  $P(c_0)$ , we have  $\sigma''_1 = \sigma''_2$

and thus  $\langle c_1, \sigma''_1 \rangle \longrightarrow \sigma'_1$  and  $\langle c_1, \sigma''_2 \rangle \longrightarrow \sigma'_2$

By inductive hypothesis  $P(c_1)$ , we then have  $\sigma'_1 = \sigma'_2$

and thus we conclude  $\sigma_1 = \sigma'_1 = \sigma'_2 = \sigma_2$

# Inductive case

$\forall b, c. P(c) \Rightarrow P(\text{while } b \text{ do } c)$       Take generic  $b, c$

We assume (inductive hypothesis)

$$P(c) \triangleq \forall \sigma, \sigma_1, \sigma_2. \langle c, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

We want to prove

$$P(\text{while } b \text{ do } c) \triangleq \forall \sigma, \sigma_1, \sigma_2. \langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Take  $\sigma, \sigma_1, \sigma_2$  such that  $\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_1$  and  $\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_2$

We want to prove  $\sigma_1 = \sigma_2$

By determinacy of boolean expressions, there are two cases

$$\langle b, \sigma \rangle \longrightarrow \mathbf{ff}$$

$$\langle b, \sigma \rangle \longrightarrow \mathbf{tt}$$

# Inductive case (ctd)

if  $\langle b, \sigma \rangle \longrightarrow \mathbf{ff}$

Consider the goal  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma_1$

Only the rule  $\frac{\langle b, \sigma \rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma}$  is applicable hence  $\sigma_1 = \sigma$

Similarly, since  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma_2$  it must be  $\sigma_2 = \sigma$   
and thus we conclude  $\sigma_1 = \sigma = \sigma_2$

# Inductive case (ctd)

if  $\langle b, \sigma \rangle \longrightarrow \mathbf{tt}$

Consider the goal  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma_1$

Only the rule 
$$\frac{\langle b, \sigma \rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma'}$$
 is applicable

hence  $\sigma_1 = \sigma'_1$  with  $\langle c, \sigma \rangle \longrightarrow \sigma''_1$  and  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma''_1 \rangle \longrightarrow \sigma'_1$

Similarly, since  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma_2$

it must be  $\sigma_2 = \sigma'_2$  with  $\langle c, \sigma \rangle \longrightarrow \sigma''_2$  and  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma''_2 \rangle \longrightarrow \sigma'_2$

By inductive hypothesis  $P(c)$ , we have  $\sigma''_1 = \sigma''_2$

thus  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma''_2 \rangle \longrightarrow \sigma'_1$  and  $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma''_2 \rangle \longrightarrow \sigma'_2$

but there is no inductive hypothesis  $P(\mathbf{while} \ b \ \mathbf{do} \ c)$  !

# Recursive definition!

one premise is as much complex as the conclusion



$$\frac{\langle b, \sigma \rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma'}$$

to close the proof of determinacy  
we need a more convenient induction principle:

Rule induction