Principles for software composition 2020/21
03 - Well founded recursion, posets and semantics

[Ex. 1] Define by well-founded recursion the function *vars* that, given an arithmetic expression a, returns the set of identifiers that appear in a. Then, prove by rule induction that $\forall a \in \mathit{Aexp}, \forall \sigma \in \Sigma, \forall n \in \mathbb{Z}$

$$\langle a, \sigma \rangle \to n \qquad \text{implies} \qquad \forall \sigma'.\, (\, (\forall y \in \mathit{vars}(a).\, \sigma(y) = \sigma'(y)) \Rightarrow \langle a, \sigma' \rangle \to n\,).$$

[Ex. 2] Define by well-founded recursion the function *vars* that, given a command, returns the set of identifiers that appear on the left-hand side of some assignment. Then, prove by rule induction that $\forall c \in \mathit{Com}, \forall \sigma, \sigma' \in \Sigma$

$$\langle c, \sigma \rangle \to \sigma' \qquad \text{implies} \qquad \forall x \notin \mathit{vars}(c).\, \sigma(x) = \sigma'(x).$$

[Ex. 3] Consider the $\mathrm{CPO}_\perp$ $(\wp(\mathbb{N}), \subseteq)$. Prove that for any set $S \subseteq \mathbb{N}$:

1. the function $f_S : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ such that $f_S(X) = X \cap S$ is continuous.
2. the function $g_S : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ such that $g_S(X) = X \cup S$ is continuous.

[Ex. 4] Prove that any limit-preserving function is monotone.

[Ex. 5] Let $D = \{n \in \mathbb{N} \mid n > 0\} \cup \{\infty\}$ and $\sqsubseteq\, \subseteq (D \times D)$ such that

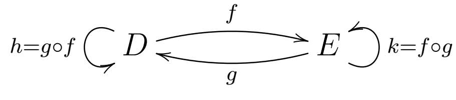
- for any $n, m \in D \cap \mathbb{N}$, we let $n \sqsubseteq m$ iff n divides m;
- for any $x \in D$, we let $x \sqsubseteq \infty$.

Is $(D, \sqsubseteq)$ a $\mathrm{CPO}_\perp$? Explain.

[Ex. 6] Define two functions $f_i : D_i \to D_i$ over two suitable CPOs D_i for $i \in [1, 2]$ (not necessarily with bottom) such that

1. f_1 is continuous, has fixpoints but not a least fixpoint;
2. f_2 is continuous and has no fixpoint;

[Ex. 7] Let D, E be two $\mathrm{CPO}_\perp$s and $f : D \to E$, $g : E \to D$ be two continuous functions between them. Their compositions $h = g \circ f : D \to D$ and $k = f \circ g : E \to E$ are known to be continuous and thus have least fixpoints.



Let $e_0 = \mathrm{fix}(k) \in E$. Prove that $g(e_0) = \mathrm{fix}(h) \in D$ by showing that

1. $g(e_0)$ is a fixpoint for h, and
2. $g(e_0)$ is the least pre-fixpoint for h.