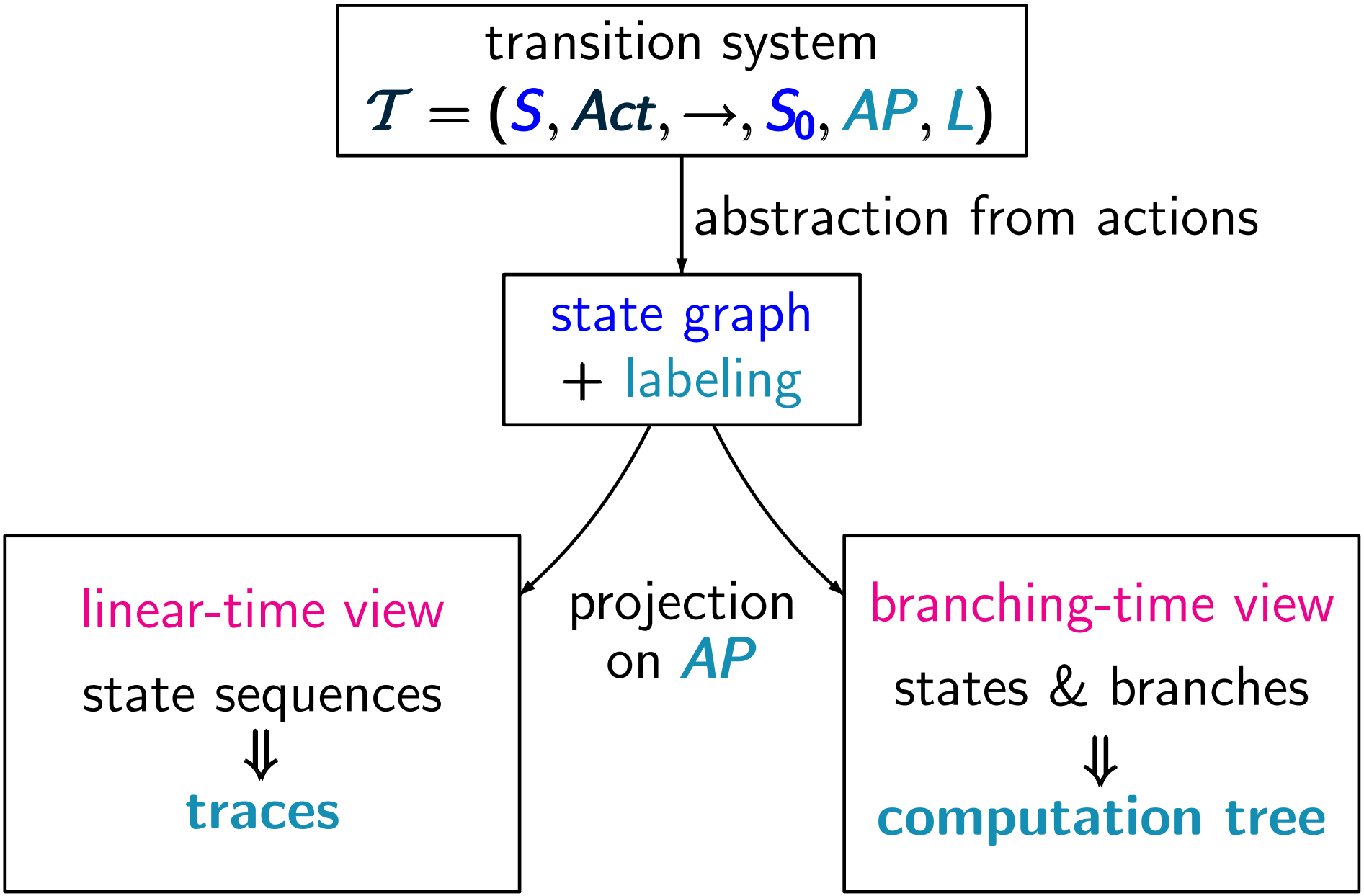
Overview

Linear vs branching time



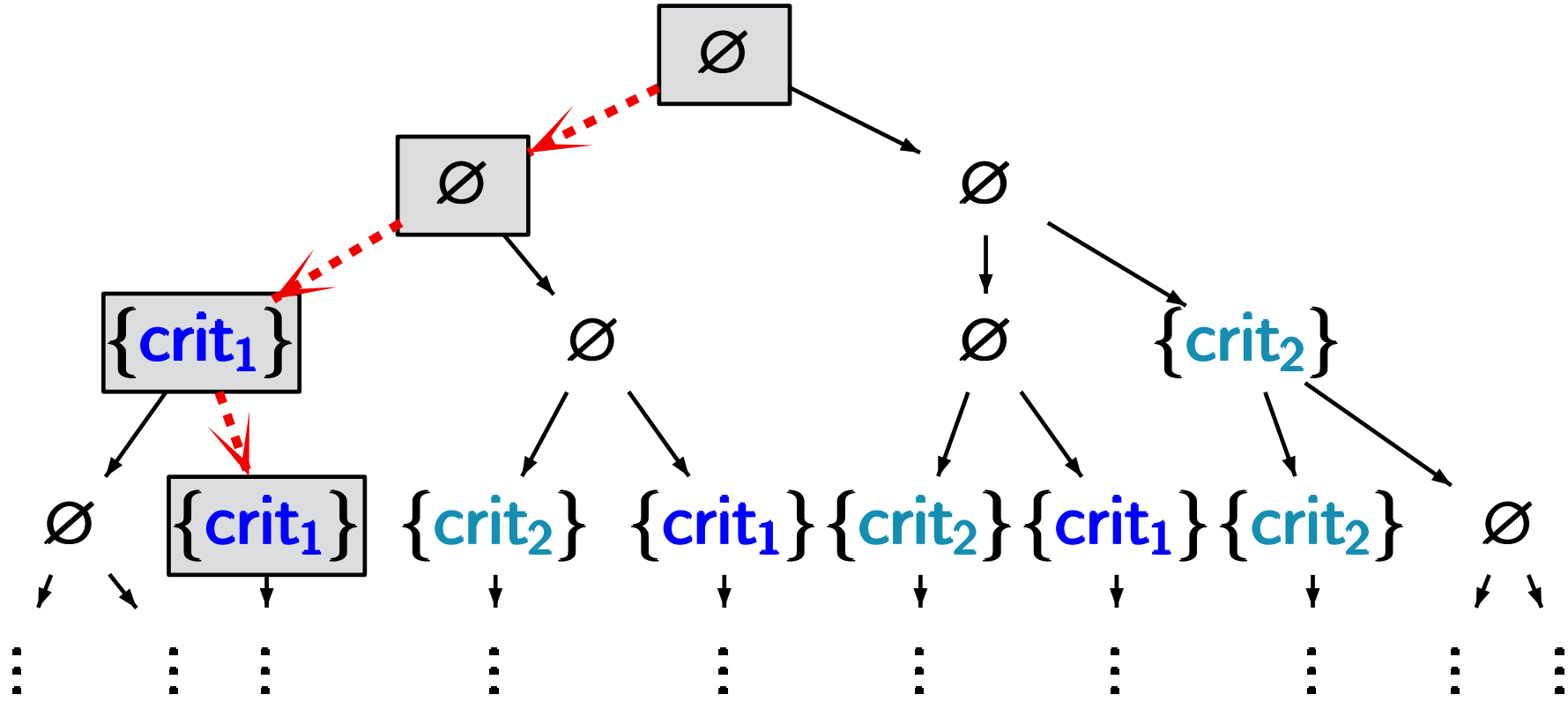
Computation tree

The computation tree of state s_0 in a transition system $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ arises by:

- unfolding $\mathcal{T}_{s_0} = (S, Act, \rightarrow, s_0, AP, L)$ into a tree
- abstraction from the actions
- projection of the states s to their labels $L(s) \subseteq AP$

Example: computation tree

mutual exclusion with semaphore and $AP = \{\mathsf{crit}_1, \mathsf{crit}_2\}$:



path	$\langle \mathsf{nc}_1, \mathsf{nc}_2 \rangle$	$\langle \mathsf{wait}_1, \mathsf{nc}_2 \rangle$	$\langle \mathsf{crit}_1, \mathsf{nc}_2 \rangle$	$\langle \mathsf{crit}_1, \mathsf{wait}_2 \rangle$	...
↓	↓	↓	↓	↓	
trace	∅	∅	$\{\mathsf{crit}_1\}$	$\{\mathsf{crit}_1\}$	...

Linear vs. branching time

	linear time	branching time
behavior	path based traces	state based computation tree
temporal logic	**LTL** path formulas	**CTL** state formulas
model checking	PSPACE-complete $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(\lvert\varphi\rvert))$	PTIME $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \lvert\Phi\rvert)$
impl. relation	trace inclusion trace equivalence PSPACE-complete	simulation bisimulation PTIME
fairness	can be encoded	requires special treatment

Computation Tree Logic (CTL)

CTL (state) formulas:

$$\Phi \quad ::= \quad \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi \mid \forall\varphi$$

CTL path formulas:

$$\varphi \quad ::= \quad \bigcirc\Phi \mid \Phi_1 \,\mathsf{U}\, \Phi_2$$

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eventually:

$$\exists\Diamond\Phi \stackrel{\text{def}}{=} \exists(\mathit{true}\,\mathsf{U}\,\Phi)$$

$$\forall\Diamond\Phi \stackrel{\text{def}}{=} \forall(\mathit{true}\,\mathsf{U}\,\Phi)$$

always:

$$\exists\Box\Phi \stackrel{\text{def}}{=} \;?$$

note: $\exists\neg\Diamond\neg\Phi$ is no **CTL** formula

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always:

$$\exists\Box\Phi \stackrel{\text{def}}{=} \neg\forall\Diamond\neg\Phi$$

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note: $\exists\neg\Diamond\neg\Phi$ is no **CTL** formula

Examples for CTL formulas

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reset possiblity $\forall\Box\exists\Diamond \mathit{start}$

unconditional process fairness $\forall\Box\forall\Diamond \mathit{crit}_1 \wedge \forall\Box\forall\Diamond \mathit{crit}_2$

Alternative (standard) Syntax for CTL

E	(∃)	there exists a path
A	(∀)	for all paths
X	( )	in the next state
U	until	(strong) (W,V,R...)
F	(<>)	finally (eventually)
G	([])	generally (always)

EFΦ: Φ holds potentially, **AFΦ**: Φ is inevitable

EGΦ: potentially always Φ, **AGΦ**: invariantly Φ

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[mutual exclusion]

- ∀[ ](~crit1 ∨ ~crit2) AG(~crit1 ∨ ~crit2)

[every request will be answered eventually]

- ∀[](request → ∀<> response) AG(request → AF response)

[traffic light]

- ∀[](yellow → ∀○red) AG(yellow → AX red)

[reset possiblity]

- ∀[]∃<> start AG EF start

Semantics of CTL

define a satisfaction relation $\models$ for CTL formulas over AP and a given TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ without terminal states

- interpretation of state formulas over the states
- interpretation of path formulas over the paths (infinite path fragments)

Recall: semantics of LTL

for infinite path fragment $\pi = s_0\, s_1\, s_2\, \ldots$:

$\pi \models \mathit{true}$

$\pi \models a$ iff $s_0 \models a$,i.e., $a \in L(s_0)$

$\pi \models \varphi_1 \wedge \varphi_2$ iff $\pi \models \varphi_1$ and $\pi \models \varphi_2$

$\pi \models \neg\varphi$ iff $\pi \not\models \varphi$

$\pi \models \bigcirc\varphi$ iff $\mathit{suffix}(\pi, 1) = s_1\, s_2\, s_3\, \ldots \models \varphi$

$\pi \models \varphi_1\, \mathsf{U}\, \varphi_2$ iff there exists $j \geq 0$ such that

$\mathit{suffix}(\pi, j) = s_j\, s_{j+1}\, s_{j+2}\, \ldots \models \varphi_2$ and

$\mathit{suffix}(\pi, k) = s_k\, s_{k+1}\, s_{k+2}\, \ldots \models \varphi_1$ for $0 \leq k < j$

Satisfaction relation for path formulas

Let $\pi = s_0\, s_1\, s_2\, \ldots$ be an infinite path fragment.

$\pi \models \bigcirc \Phi$ iff $s_1 \models \Phi$

$\pi \models \Phi_1\, \mathsf{U}\, \Phi_2$ iff there exists $j \geq 0$ such that

$s_j \models \Phi_2$

$s_k \models \Phi_1$ for $0 \leq k < j$

semantics of derived operators:

$\pi \models \Diamond \Phi$ iff there exists $j \geq 0$ with $s_j \models \Phi$

$\pi \models \Box \Phi$ iff for all $j \geq 0$ we have: $s_j \models \Phi$

Satisfaction relation for state formulas

$s \models \mathit{true}$

$s \models a$ iff $a \in L(s)$

$s \models \Phi_1 \wedge \Phi_2$ iff $s \models \Phi_1$ and $s \models \Phi_2$

$s \models \neg\Phi$ iff $s \not\models \Phi$

$s \models \exists\varphi$ iff there is a path $\pi \in \mathit{Paths}(s)$ s.t. $\pi \models \varphi$

$s \models \forall\varphi$ iff for each path $\pi \in \mathit{Paths}(s)$: $\pi \models \varphi$

satisfaction set for state formula Φ:

$$\mathit{Sat}(\Phi) \stackrel{\text{def}}{=} \{s \in S : s \models \Phi\}$$

Interpretation of CTL formulas over a TS

satisfaction of state formulas over a TS $\mathcal{T}$:

$$\mathcal{T} \models \Phi \quad \text{iff} \quad S_0 \subseteq \mathit{Sat}(\Phi)$$

$$\text{iff} \quad s_0 \models \Phi \text{ for all initial states } s_0 \text{ of } \mathcal{T}$$

where S_0 is the set of initial states

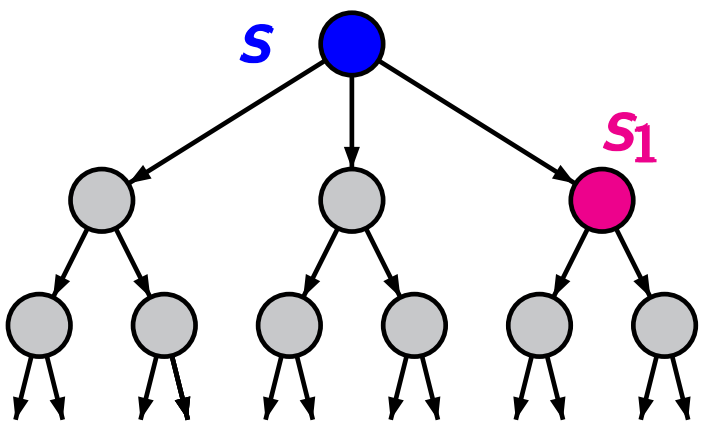
recall: $\mathit{Sat}(\Phi) = \{s \in S : s \models \Phi\}$

Semantics of the next operator

$s \models \exists\bigcirc\Phi$ iff there exists $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$ s.t. $\pi \models \bigcirc\Phi$, i.e., $s_1 \models \Phi$

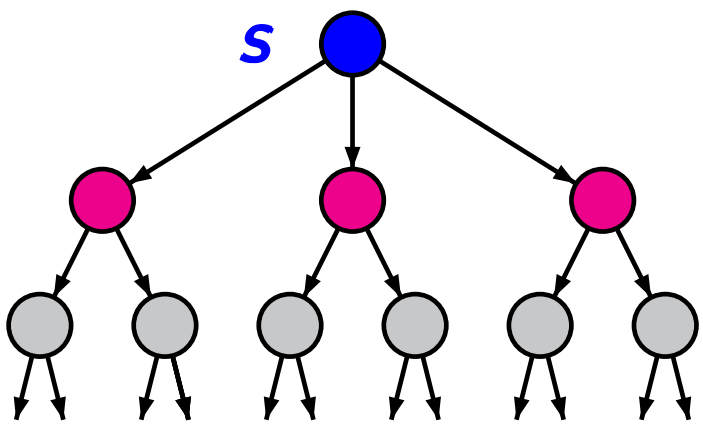
$s \models \forall\bigcirc\Phi$ iff for all $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$: $\pi \models \bigcirc\Phi$, i.e., $s_1 \models \Phi$

$\exists\bigcirc\Phi$



$Post(s) \cap Sat(\Phi) \neq \varnothing$

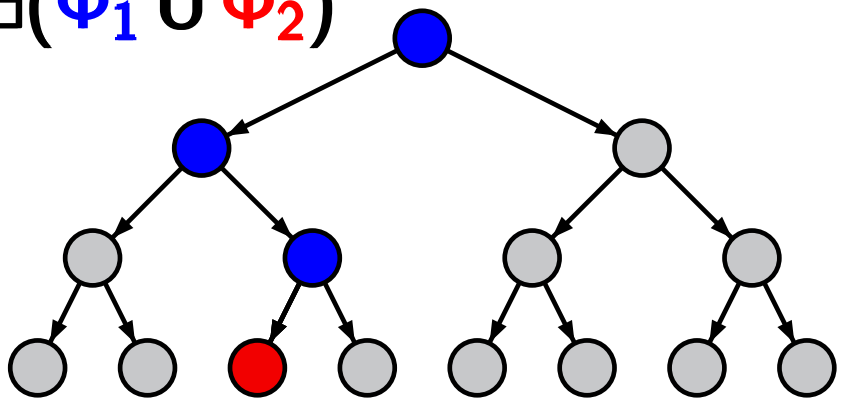
$\forall\bigcirc\Phi$



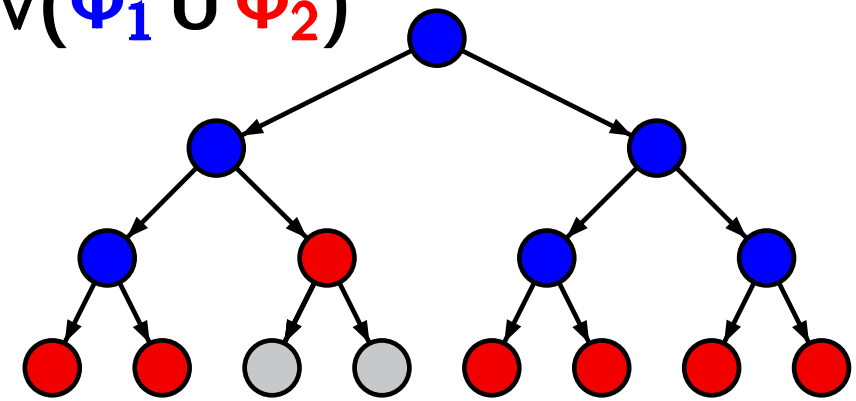
$Post(s) \subseteq Sat(\Phi)$

Semantics of until and eventually

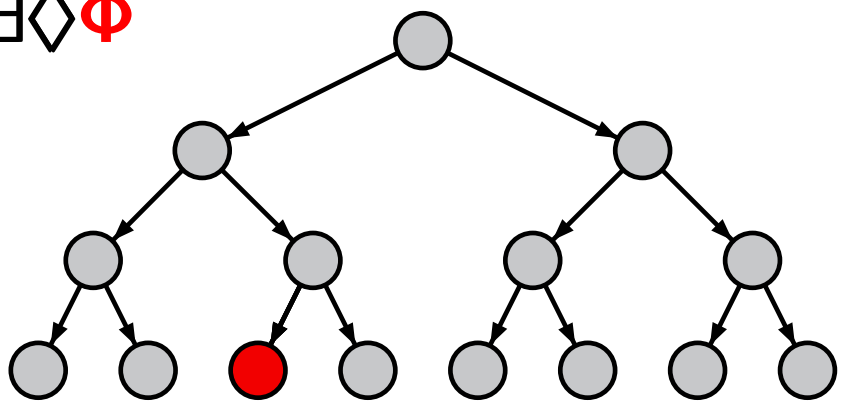
$\exists(\Phi_1 \mathsf{U} \Phi_2)$



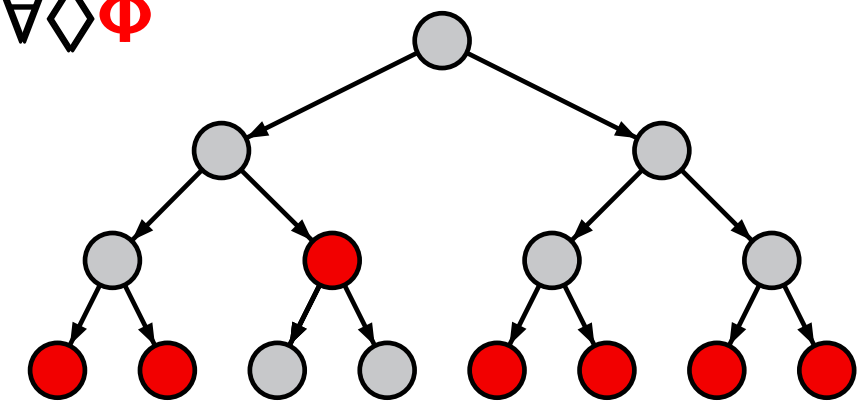
$\forall(\Phi_1 \mathsf{U} \Phi_2)$



$\exists \Diamond \Phi$

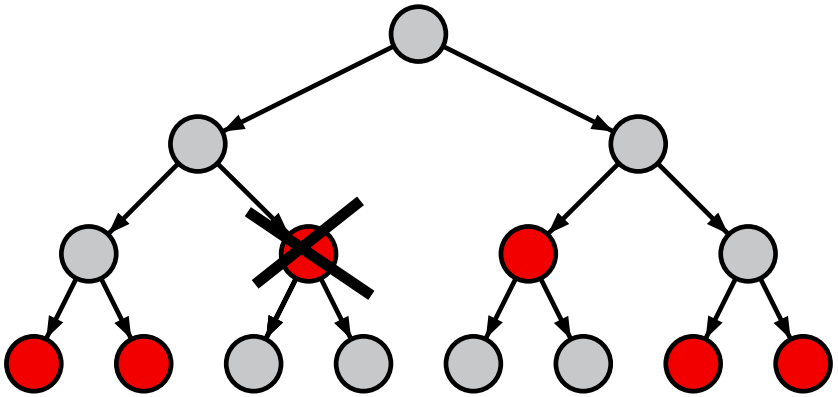


$\forall \Diamond \Phi$

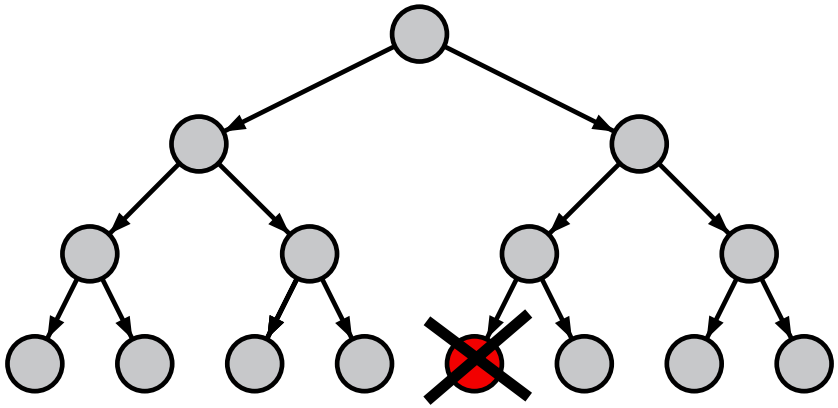


Semantics of eventually and always

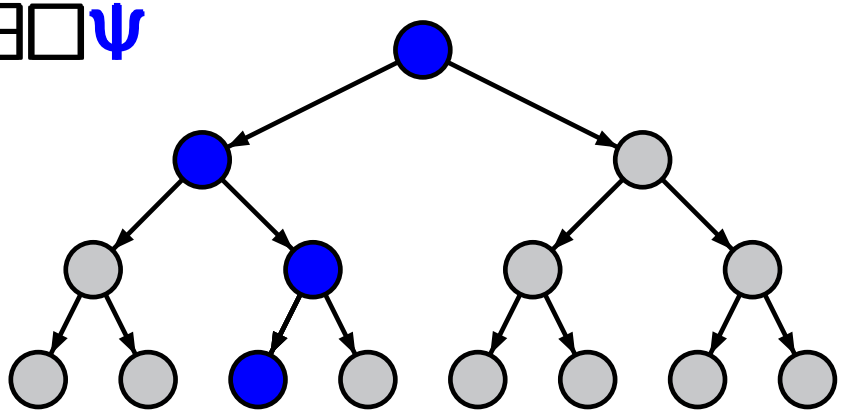
$\neg\forall\Diamond\Phi$



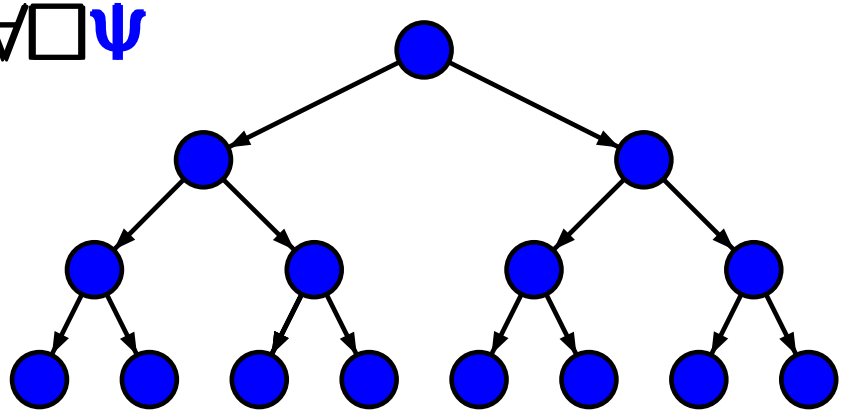
$\neg\exists\Diamond\Phi$



$\exists\Box\Psi$



$\forall\Box\Psi$



Specifying "infinitely often" in CTL

If s is a state in a TS and $a \in AP$ then:

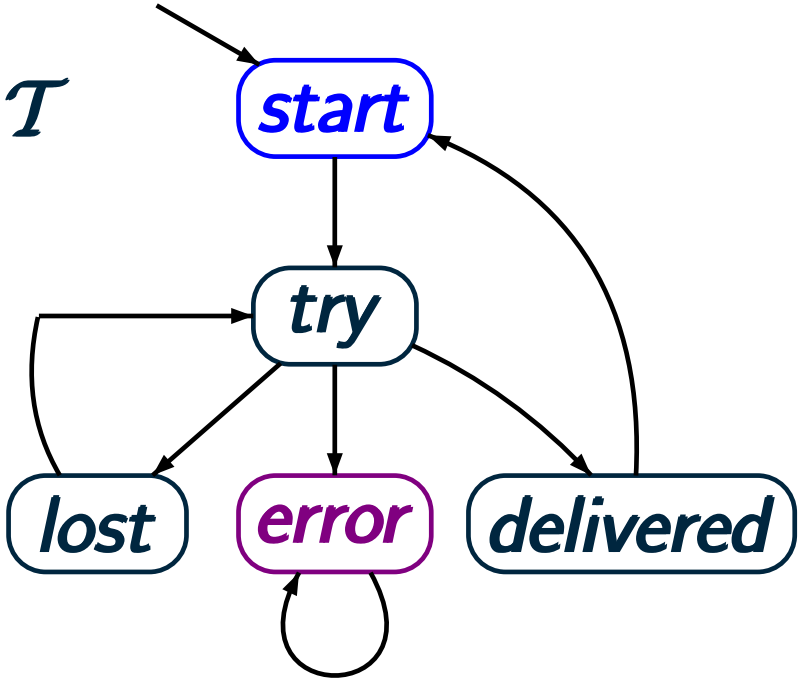
$$s \models_{CTL} \forall\Box\forall\Diamond a$$

iff for all paths $\pi = s_0\, s_1\, s_2 \ldots \in Paths(s)$:

$$\overset{\infty}{\exists}\, i \geq 0. \quad \text{s.t.} \quad s_i \models a$$

iff $$s \models_{LTL} \Box\Diamond a$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$\Phi_1 = \exists\Diamond\forall\Box\neg start$

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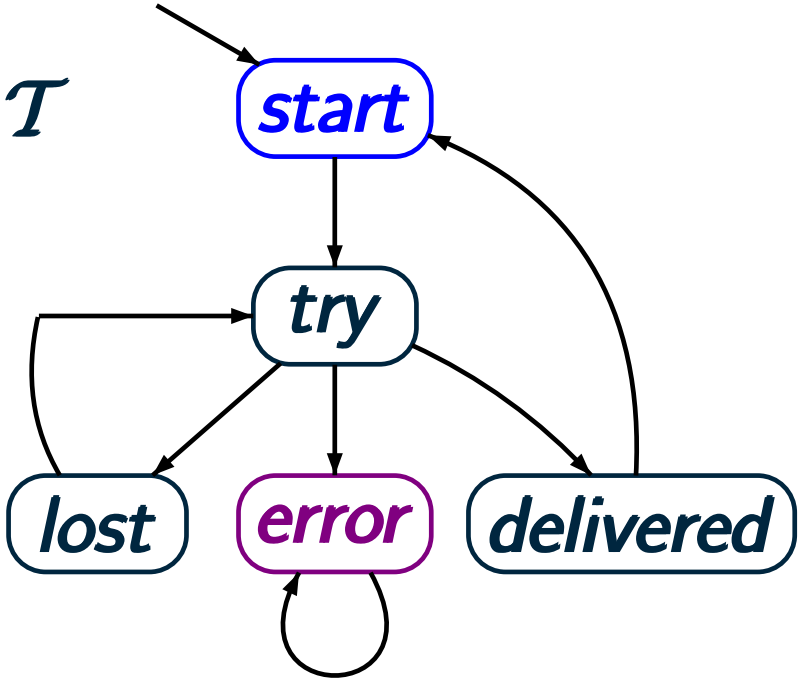
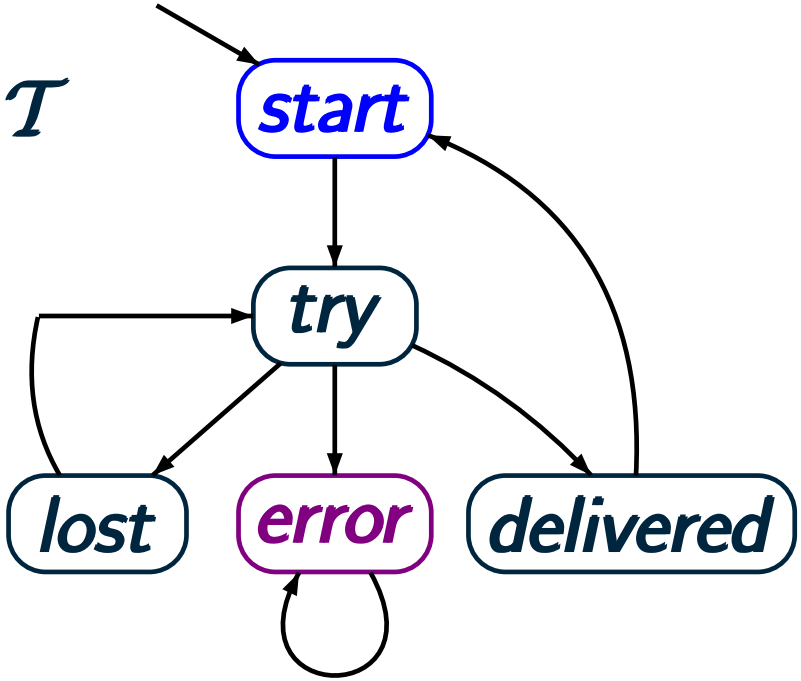


$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start}$

$Sat(\forall\Box\neg start) = \{error\}$

Example: CTL semantics



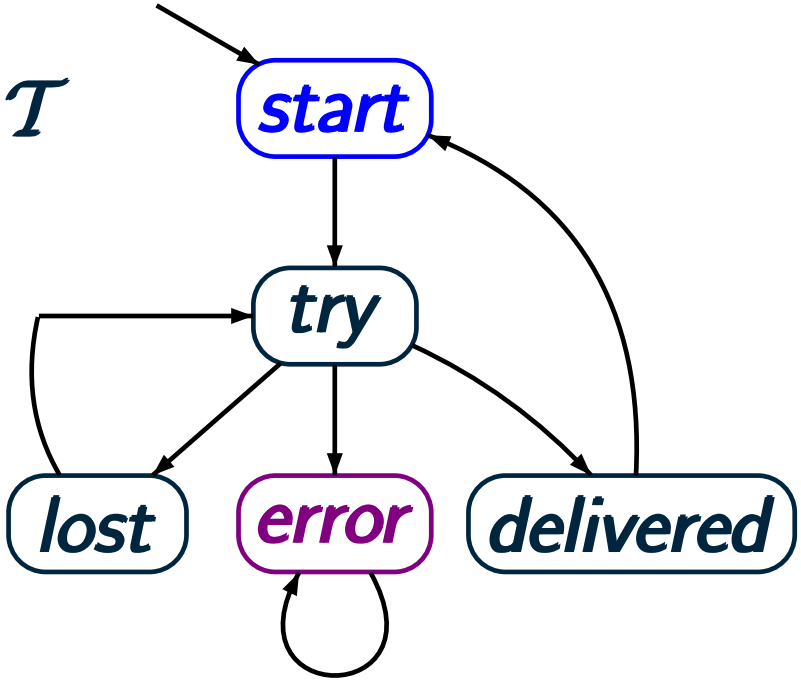
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start} \rightsquigarrow \exists\Diamond \boxed{error}$

$Sat(\forall\Box\neg start) = \{error\}$

$Sat(\exists\Diamond\forall\Box\neg start) = ?$

Example: CTL semantics



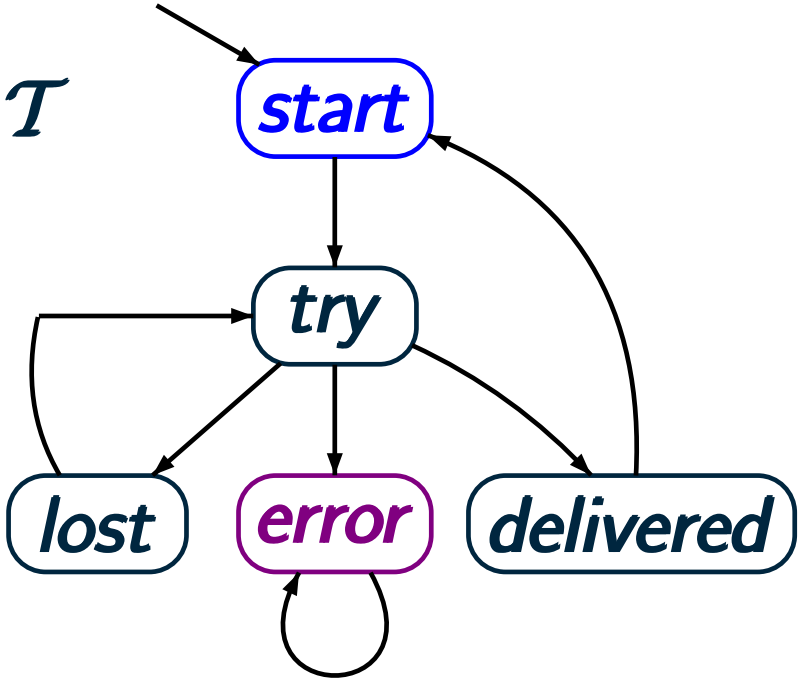
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$ $\surd$

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start} \rightsquigarrow \exists\Diamond \boxed{error}$

$Sat(\forall\Box\neg start) = \{error\}$

$Sat(\exists\Diamond\forall\Box\neg start) = Sat(\exists\Diamond error) =$ "all states"

Example: CTL semantics

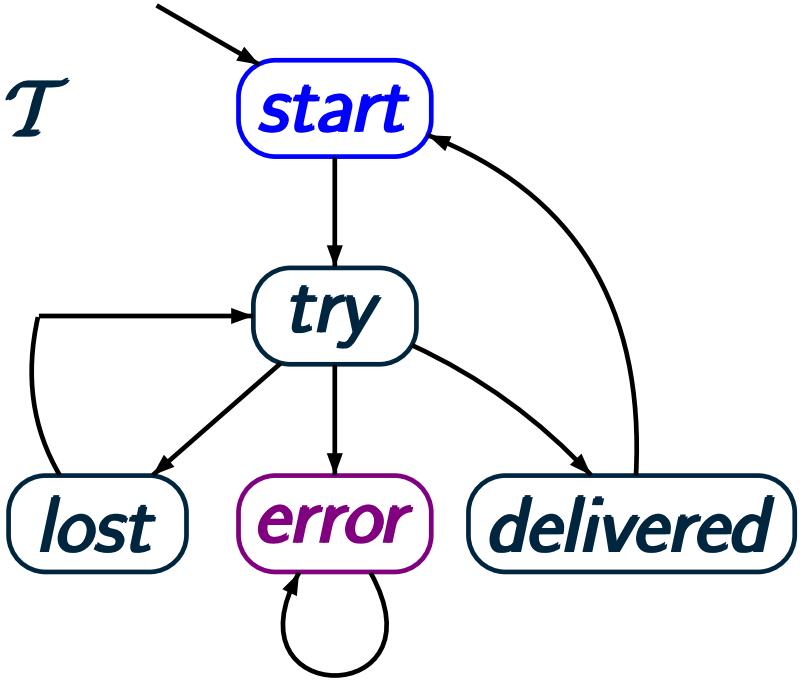


$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$?

$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

Example: CTL semantics



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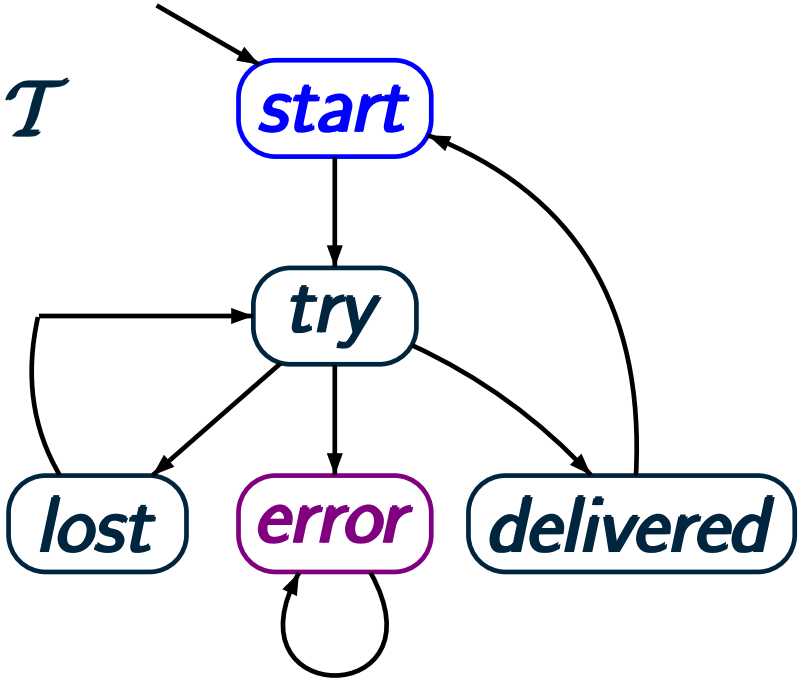

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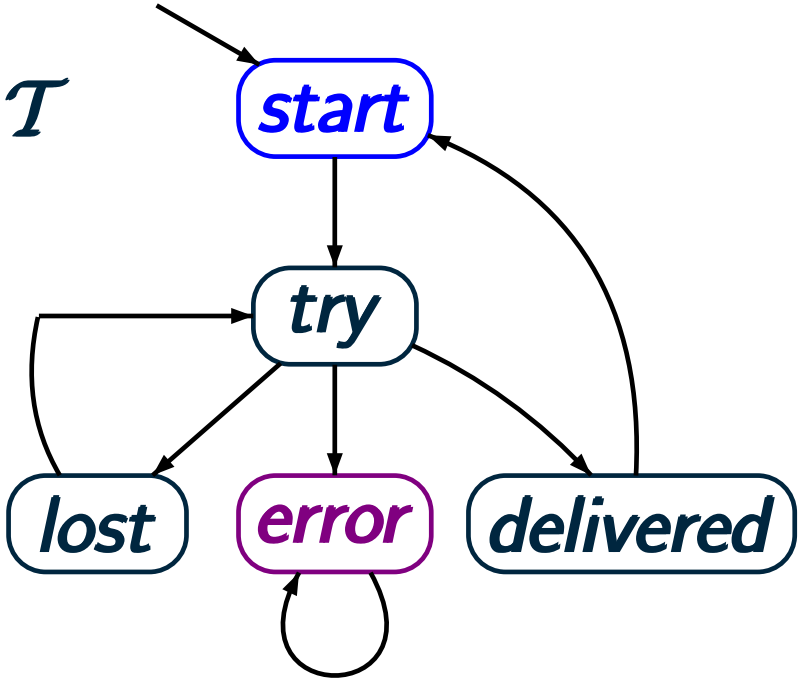
$\Phi_2 = \forall\bigcirc \boxed{\exists\bigcirc\forall\Box\neg start} \rightsquigarrow \forall\bigcirc \boxed{\exists\bigcirc error}$

$$Sat(\forall\Box\neg start) = \{error\}$$
$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$

Example: CTL semantics



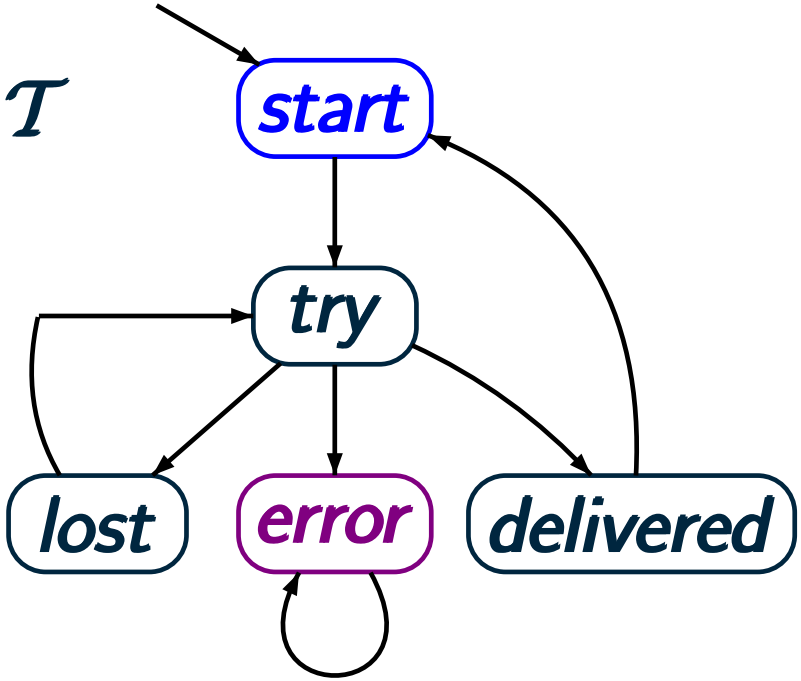
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$?

$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start \quad \rightsquigarrow \boxed{\forall\bigcirc(error \vee try)}$

$$Sat(\forall\Box\neg start) = \{error\}$$
$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$
$$Sat(\forall\bigcirc\exists\bigcirc\forall\Box\neg start) = ?$$

Example: CTL semantics



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$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ $\surd$

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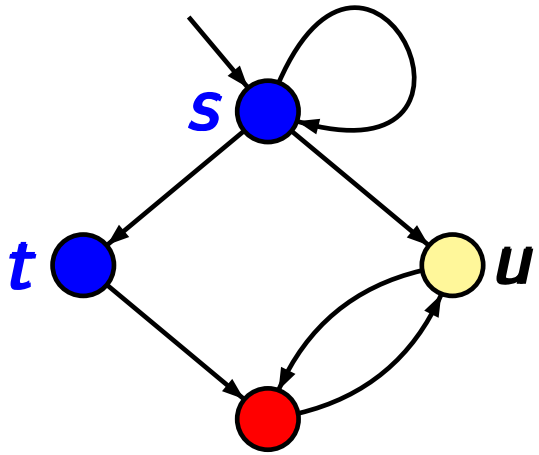
$$
\begin{aligned}
Sat(\forall\Box\neg start) &= \{error\} \\
Sat(\exists\bigcirc\forall\Box\neg start) &= \{error, try\} \\
Sat(\forall\bigcirc\exists\bigcirc\forall\Box\neg start) &= \{error, lost, start\}
\end{aligned}
$$

Example: CTL semantics



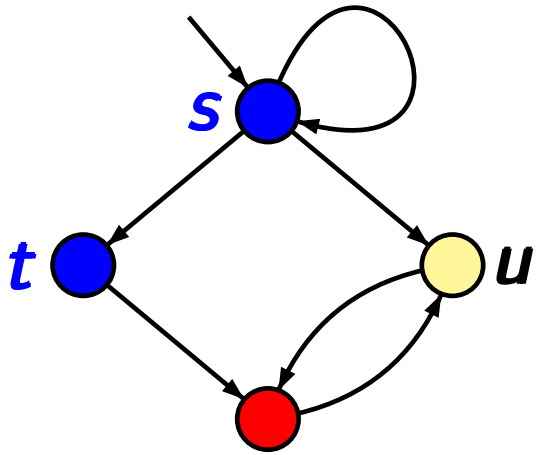
$\bullet \mathrel{\widehat{=}} \{a\}$

$\bullet \mathrel{\widehat{=}} \{b\}$

$\bullet \mathrel{\widehat{=}} \varnothing$

$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$?

Example: CTL semantics



● $\widehat{=} \{a\}$

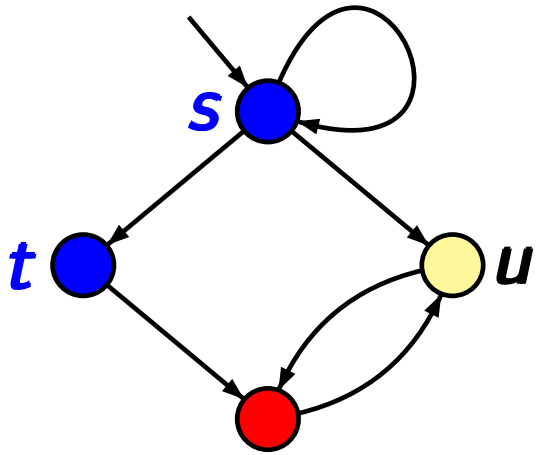
● $\widehat{=} \{b\}$

● $\widehat{=} \varnothing$

$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$ ✓ as $s\,s\,s\,\ldots \models \Box\exists(a \,\mathsf{U}\, b)$

$\mathcal{T} \models \exists((\exists\bigcirc a) \,\mathsf{U}\, b)$?

Example: CTL semantics



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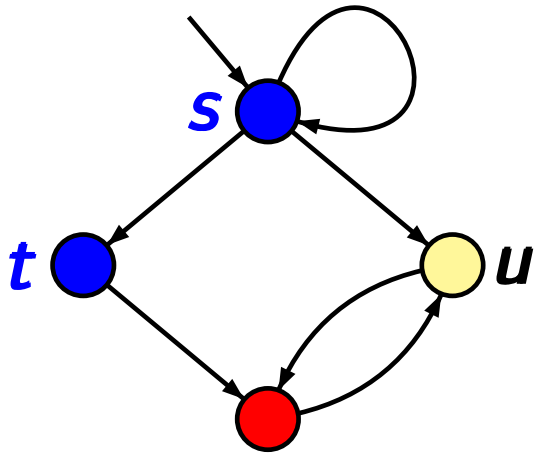
$\bullet \mathrel{\widehat{=}} \varnothing$

$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$ $\surd$ as $s\,s\,s\,\ldots \models \Box\exists(a \,\mathsf{U}\, b)$

$\mathcal{T} \not\models \exists((\exists\bigcirc a) \,\mathsf{U}\, b)$ as $t \not\models \exists\bigcirc a$, $u \not\models \exists\bigcirc a$

$\mathcal{T} \models \exists(a \,\mathsf{U}\, \forall(\neg a \,\mathsf{U}\, b))$?

Example: CTL semantics



$\bullet \mathrel{\widehat{=}} \{a\}$

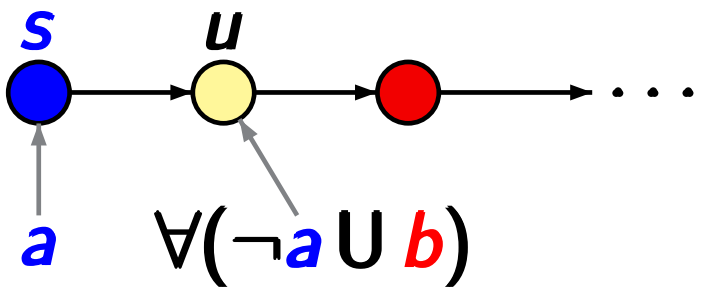
$\bullet \mathrel{\widehat{=}} \{b\}$

$\bullet \mathrel{\widehat{=}} \varnothing$

$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$ $\surd$ as $s\,s\,s\,\ldots \models \Box\exists(a \,\mathsf{U}\, b)$

$\mathcal{T} \not\models \exists((\exists\bigcirc a) \,\mathsf{U}\, b)$ as $t \not\models \exists\bigcirc a$, $u \not\models \exists\bigcirc a$

$\mathcal{T} \models \exists(a \,\mathsf{U}\, \forall(\neg a \,\mathsf{U}\, b))$ $\surd$



$\models a \,\mathsf{U}\, \forall(\neg a \,\mathsf{U}\, b)$

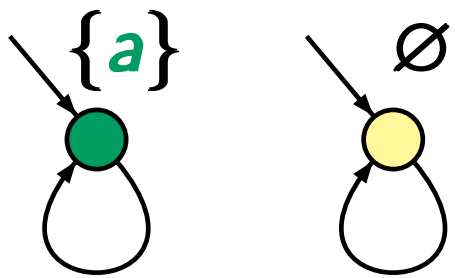
Correct or wrong?

Let $\mathcal{T}$ be a transition system and Φ a CTL formula.
Is the following statement correct ?

$$\text{if } \mathcal{T} \not\models \neg\Phi \text{ then } \mathcal{T} \models \Phi$$

answer: no

transition system $\mathcal{T}$ with 2 initial states:



$\mathcal{T} \not\models \exists\Box a$

$\mathcal{T} \not\models \neg\exists\Box a$

Equivalence of CTL formulas

$\Phi_1 \equiv \Phi_2$ iff for all transition systems $\mathcal{T}$:

$$\mathcal{T} \models \Phi_1 \iff \mathcal{T} \models \Phi_2$$

iff for all transition systems $\mathcal{T}$:

$$Sat(\Phi_1) = Sat(\Phi_2)$$

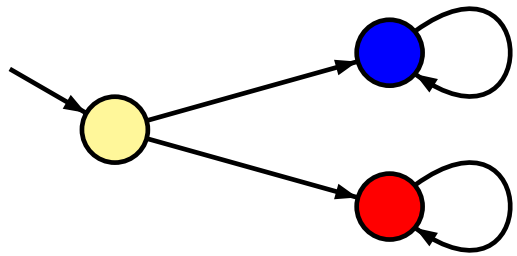
Examples:

$$\begin{aligned} \neg\neg\Phi &\equiv \Phi \\ \neg(\Phi \wedge \Psi) &\equiv \neg\Phi \vee \neg\Psi \\ &\vdots \\ \neg\forall\bigcirc\Phi &\equiv \exists\bigcirc\neg\Phi \end{aligned}$$

Correct or wrong?

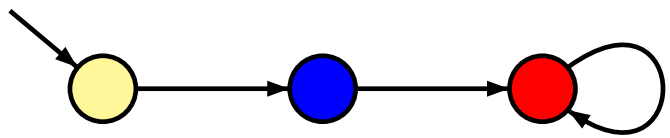
$$\exists\Diamond(a \wedge b) \equiv \exists\Diamond a \wedge \exists\Diamond b$$

wrong, e.g,



$$\forall\Diamond(a \wedge b) \equiv \forall\Diamond a \wedge \forall\Diamond b$$

wrong, e.g.,



but:

$$\forall\Box(\Phi_1 \wedge \Phi_2) \equiv \forall\Box\Phi_1 \wedge \forall\Box\Phi_2$$

$$\exists\Diamond(\Phi_1 \vee \Phi_2) \equiv \exists\Diamond\Phi_1 \vee \exists\Diamond\Phi_2$$

Expansion laws

$$\exists(\phi \,\mathsf{U}\, \psi) \equiv \psi \vee (\phi \wedge \exists\bigcirc\exists(\phi \,\mathsf{U}\, \psi))$$

$$\forall(\phi \,\mathsf{U}\, \psi) \equiv \psi \vee (\phi \wedge \forall\bigcirc\forall(\phi \,\mathsf{U}\, \psi))$$

$$\exists\Diamond\psi \equiv \psi \vee \exists\bigcirc\exists\Diamond\psi$$

$$\forall\Diamond\psi \equiv \psi \vee \forall\bigcirc\forall\Diamond\psi$$

$$\exists(\phi \,\mathsf{W}\, \psi) \equiv \psi \vee (\phi \wedge \exists\bigcirc\exists(\phi \,\mathsf{W}\, \psi))$$

$$\forall(\phi \,\mathsf{W}\, \psi) \equiv \psi \vee (\phi \wedge \forall\bigcirc\forall(\phi \,\mathsf{W}\, \psi))$$

$$\exists\Box\phi \equiv \phi \wedge \exists\bigcirc\exists\Box\phi$$

$$\forall\Box\phi \equiv \phi \wedge \forall\bigcirc\forall\Box\phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of $\mathbf{U}$ and $\mathbf{W}$, e.g.:

$$\begin{aligned}\forall(\Phi\,\mathbf{U}\,\Psi) &\equiv \neg\exists((\Phi\wedge\neg\Psi)\,\mathbf{W}\,(\neg\Phi\wedge\neg\Psi))\\ &\equiv \neg\exists((\neg\Psi)\,\mathbf{W}\,(\neg\Phi\wedge\neg\Psi))\\ &\equiv \neg\exists((\neg\Psi)\,\mathbf{U}(\neg\Phi\wedge\neg\Psi)) \wedge \neg\exists\Box\neg\Psi\end{aligned}$$

Existential normalform for CTL

For each **CTL** formula Ψ there is an equivalent **CTL** formula Φ built by

- operators of propositional logic
- the modalities $\exists\bigcirc$, $\exists\mathbf{U}$ and $\exists\Box$.

$$\Phi ::= \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\bigcirc\Phi \mid \exists(\Phi_1 \,\mathbf{U}\, \Phi_2) \mid \exists\Box\Phi$$

transformation $\Psi \rightsquigarrow \Phi$ relies on:

$$\forall\bigcirc\Psi \rightsquigarrow \neg\exists\bigcirc\neg\Psi$$

$$\forall(\Psi_1 \,\mathbf{U}\, \Psi_2) \rightsquigarrow \neg\exists(\neg\Psi_2 \,\mathbf{U}(\neg\Psi_1 \wedge \neg\Psi_2)) \wedge \neg\exists\Box\neg\Psi_2$$

Overview

Equivalence of CTL and LTL formulas

Let Φ be a **CTL** formula and φ an **LTL** formula.

$\Phi \equiv \varphi$ iff for all transition systems $\mathcal{T}$ and all states s in $\mathcal{T}$:

$$s \models_{\mathsf{CTL}} \Phi \iff s \models_{\mathsf{LTL}} \varphi$$

e.g.,

CTL formula Φ	LTL formula φ
a	a
$\forall \bigcirc a$	$\bigcirc a$
$\forall (a \,\mathsf{U}\, b)$	$a \,\mathsf{U}\, b$

$a, b \in AP$

More examples

CTL formula Φ	LTL formula φ
a	a
$\forall \bigcirc a$	$\bigcirc a$
$\forall (a \,\mathsf{U}\, b)$	$a \,\mathsf{U}\, b$
$\forall \Box a$	$\Box a$
$\forall \Diamond a$	$\Diamond a$
$\forall (a \,\mathsf{W}\, b)$	$a \,\mathsf{W}\, b$
$\forall \Box \forall \Diamond a$	$\Box \Diamond a$

infinitely often a

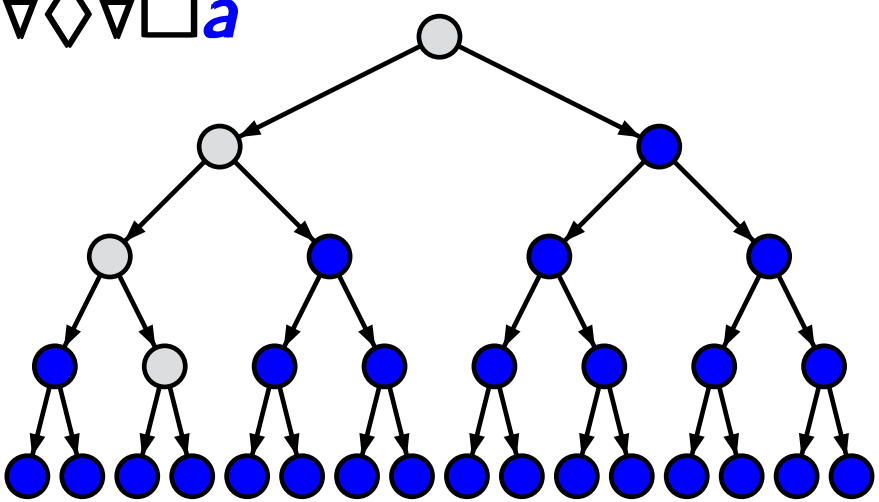
but: $\forall \Diamond \forall \Box a \not\equiv \Diamond \Box a$

The CTL formula $\forall\Diamond\forall\Box a$

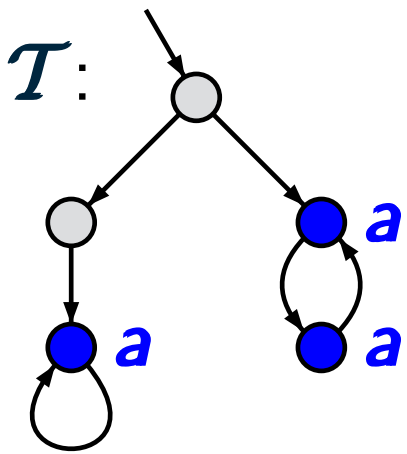
$s \models \forall\Diamond\forall\Box a$ iff on each path π from s there is a state t with $t \models \forall\Box a$

i.e., all states in the computation tree of t fulfill a

$\forall\Diamond\forall\Box a$

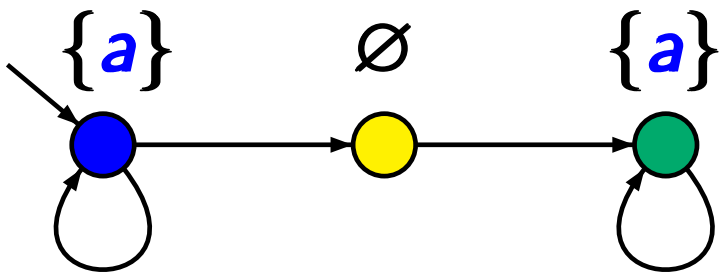


$\mathcal{T}$:



$\mathcal{T} \models \forall\Diamond\forall\Box a$

transition system $\mathcal{T}$

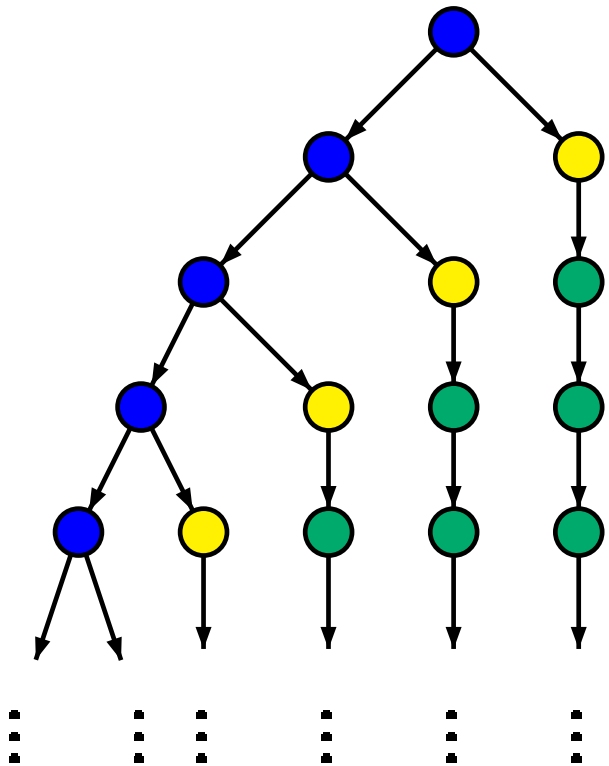


$\mathcal{T} \models_{\text{LTL}} \Diamond\Box a$

$\mathcal{T} \not\models_{\text{CTL}} \forall\Diamond\forall\Box a$

$Sat(\forall\Box a) = \{\bullet\}$

computation tree



From CTL to LTL, if possible

For each **CTL** formula Φ the following holds:

- either there is **no** equivalent LTL formula
- or $\Phi \equiv \varphi$

  where φ is the **LTL** formula obtained from Φ by removing of all path quantifiers $\exists$ and $\forall$

without proof

$\Phi = \forall\Diamond\forall\Box a$

$\downarrow$

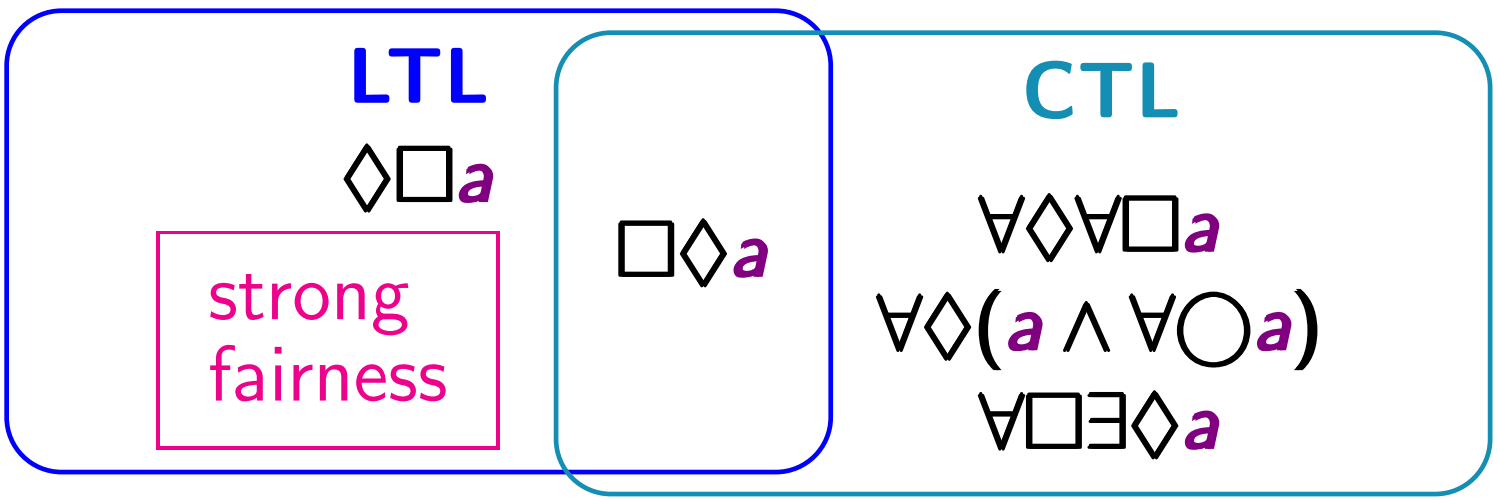
$\varphi = \Diamond\Box a \not\equiv \Phi$

hence: there is no LTL formula equivalent to Φ

Expressiveness of LTL and CTL

The expressive powers of **LTL** and **CTL** are incomparable

- The **CTL** formulas $\forall\Diamond(a \wedge \forall\bigcirc a)$, $\forall\Diamond\forall\Box a$ and $\forall\Box\exists\Diamond a$ have no equivalent LTL formula
- The **LTL** formula $\Diamond\Box a$ has no equivalent CTL formula



LTL formula $\Diamond\Box a$

There is no **CTL** formula which is equivalent to the **LTL** formula $\Diamond\Box a$

Proof (sketch): provide sequences $(\mathcal{T}_n)_{n\geq 0}$, $(\mathcal{T}'_n)_{n\geq 0}$ of transition systems such that for all $n \geq 0$:

(1) $\mathcal{T}_n \not\models \Diamond\Box a$

(2) $\mathcal{T}'_n \models \Diamond\Box a$

(3) $\mathcal{T}_n$ and $\mathcal{T}'_n$ satisfy the same **CTL** formulas length $\leq n$

Overview

CTL model checking

given: finite TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
CTL formula Φ over AP

question: does $\mathcal{T} \models \Phi$ hold ?

idea:

- compute $Sat(\Phi) = \{s \in S : s \models \Phi\}$
- check whether $S_0 \subseteq Sat(\Phi)$

CTL model checking

given: finite TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
CTL formula Φ over AP

question: does $\mathcal{T} \models \Phi$ hold ?

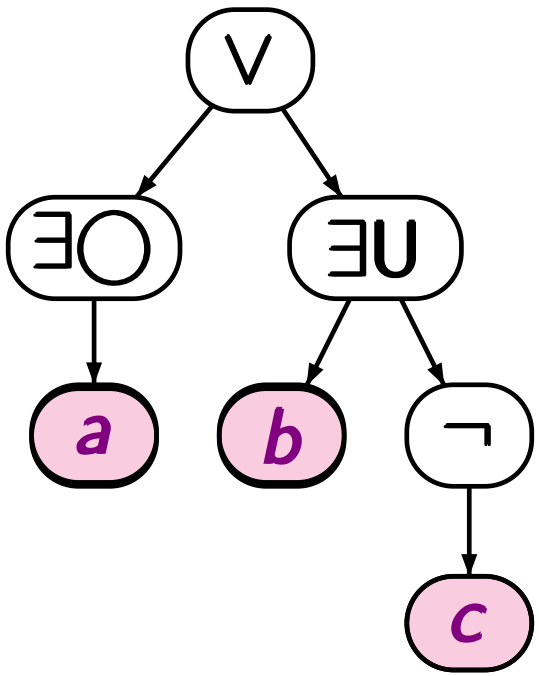
inner subformulas first

```
FOR ALL subformulas Ψ of Φ DO
    compute Sat(Ψ)
    replace Ψ by a new atomic proposition a_Ψ
    FOR ALL s ∈ Sat(Ψ) DO add a_Ψ to L(s) OD
OD
IF S0 ⊆ Sat(Φ) THEN output "yes"
               ELSE output "no"
FI
```

Example: CTL model checking

$$\Phi = \quad \exists\bigcirc a \ \vee \exists(b \,\mathsf{U}\, \neg c)$$

syntax tree for Φ



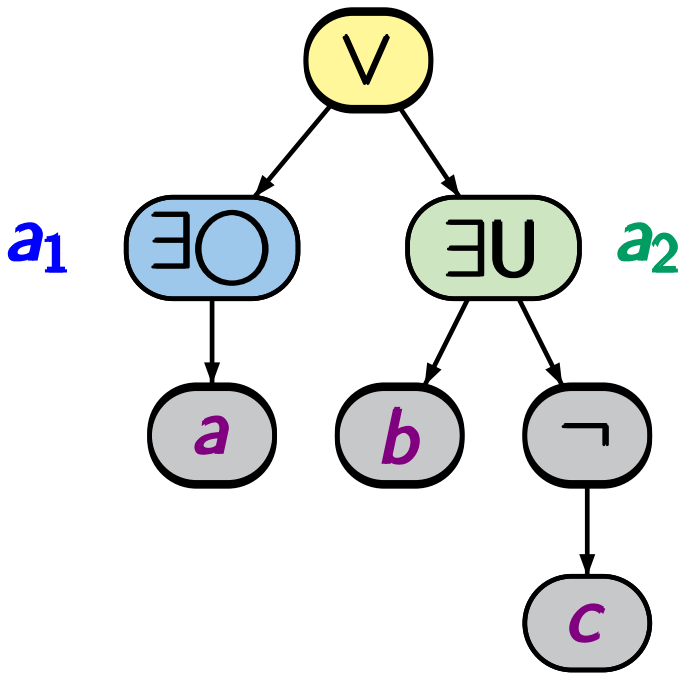
compute $Sat(a), Sat(b), Sat(c)$

processed in
bottom-up fashion

Example: CTL model checking

$$\Phi = \underbrace{\exists\bigcirc a}_{\Phi_1} \vee \underbrace{\exists(b \,\mathsf{U}\, \neg c)}_{\Phi_2} \quad \rightsquigarrow \quad a_1 \vee a_2$$

syntax tree for Φ



processed in bottom-up fashion

compute $Sat(a), Sat(b), Sat(c)$

$Sat(\Phi_1) = \ldots = Sat(a_1)$

$Sat(\neg c) = S \setminus Sat(c)$

$Sat(\Phi_2) = \ldots = Sat(a_2)$

replace Φ_1 with a_1

replace Φ_2 with a_2

$Sat(\Phi) = Sat(a_1) \cup Sat(a_2)$

CTL model checking

given: finite TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
CTL formula Φ over AP

question: does $\mathcal{T} \models \Phi$ hold ?

method: regard in bottom-up manner all subformulas Ψ of Φ and compute their satisfaction sets

$$Sat(\Psi) = \{s \in S : s \models \Psi\}$$

here: explanations for the case that Φ is in existential normal form

analogous algorithms can be designed for standard CTL (and the derived operators)

Recall: Existential normal form for CTL

For each **CTL** formula there is an equivalent formula in **∃-normal form**, i.e., a **CTL** formula with the basis modalities $\exists\bigcirc$, $\exists\mathbf{U}$, $\exists\Box$.

CTL formulas in ∃-normal form:

$$\Psi ::= \mathit{true} \mid a \mid \neg\Psi \mid \Psi_1 \wedge \Psi_2 \mid \exists\bigcirc\Psi \mid \exists(\Psi_1 \,\mathbf{U}\, \Psi_2) \mid \exists\Box\Psi$$

CTL formula $\rightsquigarrow$ **CTL** formula in ∃-normal form

$$\forall\bigcirc\Phi \rightsquigarrow \neg\exists\bigcirc\neg\Phi$$

$$\forall(\Phi_1 \,\mathbf{U}\, \Phi_2) \rightsquigarrow \neg\exists(\neg\Phi_2 \,\mathbf{U}(\neg\Phi_1 \wedge \neg\Phi_2)) \wedge \neg\exists\Box\neg\Phi_2$$

Recursive computation of the satisfaction sets

$$
\begin{aligned}
Sat(true) &= S \\
Sat(a) &= \{s \in S : a \in L(s)\} \\
Sat(\neg\Phi) &= S \setminus Sat(\Phi) \\
Sat(\Phi_1 \wedge \Phi_2) &= Sat(\Phi_1) \cap Sat(\Phi_2) \\
Sat(\exists \bigcirc \Phi) &= \{s \in S : Post(s) \cap Sat(\Phi) \neq \varnothing\} \\
Sat(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)) &= \ldots \\
Sat(\exists \Box \Phi) &= \ldots
\end{aligned}
$$

treatment of $\exists \mathsf{U}$ and $\exists \Box$:

via fixed point computation

Least fixed point characterization of $\exists \mathsf{U}$

$$\exists(\Phi_1 \,\mathsf{U}\, \Phi_2) \;\equiv\; \Phi_2 \vee \big(\Phi_1 \wedge \exists\bigcirc\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)\big)$$

$$\mathit{Sat}(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)) = \mathit{Sat}(\Phi_2) \cup \big\{ s \in \mathit{Sat}(\Phi_1) : \mathit{Post}(s) \cap \mathit{Sat}(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)) \neq \varnothing \big\}$$

satisfies the following conditions:

(1) $\mathit{Sat}(\Phi_2) \subseteq \mathit{Sat}(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2))$

(2) If $s \in \mathit{Sat}(\Phi_1)$ and $\mathit{Post}(s) \cap \mathit{Sat}(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)) \neq \varnothing$ then $s \in \mathit{Sat}(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2))$

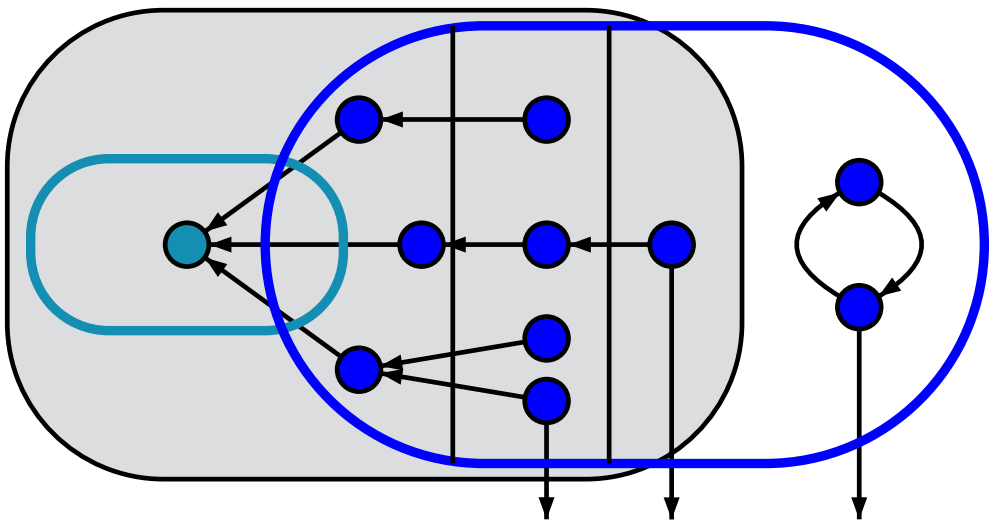
$\mathit{Sat}(\exists(\Phi_1 \mathsf{U} \Phi_2))$ is the smallest set s.t. (1) and (2) hold

CTL model checking: until operator

$$\exists(\Phi_1 \,\mathsf{U}\, \Phi_2) \;\equiv\; \Phi_2 \vee (\Phi_1 \wedge \exists\bigcirc\exists(\Phi_1 \,\mathsf{U}\, \Phi_2))$$

$Sat(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)) =$ least set T of states s.t.

$$Sat(\Phi_2) \cup \{s \in Sat(\Phi_1) : Post(s) \cap T \neq \varnothing\} \subseteq T$$



$Sat(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2))$

CTL model checking: always operator

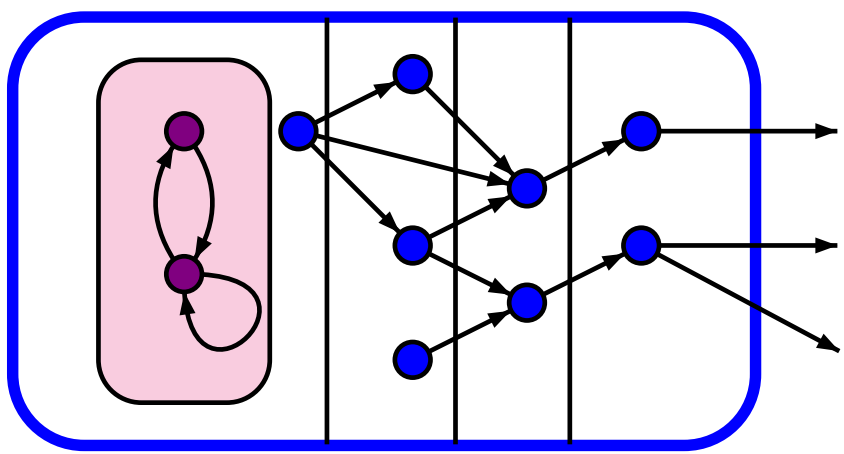
expansion law: $\exists\Box\Phi \equiv \Phi \wedge \exists\bigcirc\exists\Box\Phi$

$Sat(\exists\Box\Phi)$ = greatest set T of states with

$$T \subseteq \{s \in Sat(\Phi) : Post(s) \cap T \neq \varnothing\}$$

$$T_0 := Sat(\Phi), \quad T_{n+1} := \{s \in T_n : Post(s) \cap T_n \neq \varnothing\}$$

$Sat(\Phi)$



Recursive computation of $Sat(\ldots)$

$$Sat(\Phi_1 \wedge \Phi_2) = Sat(\Phi_1) \cap Sat(\Phi_2)$$

$$Sat(\neg\Phi) = S \setminus Sat(\Phi)$$

$$Sat(\exists\bigcirc\Phi) = \{s \in S : Post(s) \cap Sat(\Phi) = \varnothing\}$$

$Sat(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2))$ = smallest set T of states s.t.

- $Sat(\Phi_2) \subseteq T$
- $s \in Sat(\Phi_1)$ and $Post(s) \cap T \neq \varnothing \implies s \in T$

$Sat(\exists\Box\Phi)$ = greatest set V of states s.t.

- $V \subseteq Sat(\Phi)$
- $s \in V \implies Post(s) \cap V \neq \varnothing$

CTL model checking: treatment of $\exists U$

compute $Sat(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2))$ via an enumerative backward search

$T := Sat(\Phi_2)$ ⟵ collects all states $s \models \exists(\Phi_1 \,\mathsf{U}\, \Phi_2)$

$E := Sat(\Phi_2)$ ⟵ set of states still to be expanded

```
WHILE E ≠ ∅ DO
    select a state s' ∈ E and remove s' from E
    FOR ALL s ∈ Pre(s') DO
        IF s ∈ Sat(Φ1) \ T THEN add s to T and E FI
    OD
OD
return T
```

complexity: $\mathcal{O}(size(\mathcal{T}))$

Computation of $Sat(\exists\Box\Phi)$

$T := Sat(\Phi)$ ← organizes the candidates for $s \models \exists\Box\Phi$

$E := S \setminus T$ ← set of states to be expanded

```
WHILE E ≠ ∅ DO
    pick a state s' ∈ E and remove s' from E
    FOR ALL s ∈ Pre(s') DO
      IF s ∈ T and Post(s) ∩ T = ∅ THEN
          remove s from T and add s to E
      FI
OD
return T
```

naïve implementation:
quadratic time complexity

Computation of $Sat(\exists\Box\Phi)$ using counters

$T := Sat(\Phi)$; $E := S \setminus T$

FOR ALL $s \in Sat(\Phi)$ DO $c[s] := |Post(s)|$ OD

loop invariant: $c[s] = |Post(s) \cap (T \cup E)|$ for $s \in T$

WHILE $E \neq \emptyset$ DO

pick a state $s' \in E$ and remove s' from E

FOR ALL $s \in Pre(s')$ DO

IF $s \in T$ THEN

$c[s] := c[s] - 1$

IF $c[s] = 0$ THEN

remove s from T and add s to E FI

FI

OD

complexity:
$\mathcal{O}(size(\mathcal{T}))$

Recursive computation of $Sat(\ldots)$

$$Sat(\Phi_1 \wedge \Phi_2) = Sat(\Phi_1) \cap Sat(\Phi_2)$$

$$Sat(\neg\Phi) = S \setminus Sat(\Phi)$$

$$Sat(\exists\bigcirc\Phi) = \{s \in S : Post(s) \cap Sat(\Phi) = \varnothing\}$$

$$Sat(\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)) = \bigcup_{n\geq 0} T_n \text{ where}$$

$$T_0 = Sat(\Phi_2)$$

$$T_{n+1} = \{s \in Sat(\Phi_1) : Post(s) \cap T_n \neq \varnothing\}$$

$$Sat(\exists\Box\Phi) = \bigcap_{n\geq 0} V_n \text{ where}$$

$$V_0 = Sat(\Phi);\quad V_{n+1} = \{s \in V_n : Post(s) \cap V_n \neq \varnothing\}$$

Complexity of CTL/LTL model checking

CTL model checking: $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot |\Phi|)$

LTL model checking: $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(|\varphi|))$

model complexity, i.e., for fixed specification:

CTL and **LTL**: $\mathcal{O}(\mathit{size}(\mathcal{T}))$

If $\Phi \equiv \varphi$ then "often" we have: $|\Phi| = \exp(|\varphi|)$

CTL vs LTL

If $P \neq NP$ then there is a sequence $(\varphi_n)_{n \geq 0}$ of **LTL** formulas such that:

- $|\varphi_n| = \mathcal{O}(poly(n))$
- φ_n has an equivalent **CTL** formula, but no equivalent **CTL** formula of polynomial length

Proof. Let $\varphi_n = \neg\varphi'_n$. Then:

- $|\varphi_n| = |\varphi'_n| + 1 = \mathcal{O}(n^2)$
- φ_n has an equivalent **CTL** formula, e.g., $\neg\Phi'_n$

Suppose there is a **CTL** formula of polynomial length that is equivalent to φ_n. Then:

Hamilton path problem $\in P$ and $P = NP$

Overview

Complexity of CTL and LTL model checking

LTL model checking problem:

PSPACE-complete and solvable in time

$$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(|\varphi|))$$

CTL model checking problem:

solvable in polynomial time (even PTIME-complete)

$$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot |\Phi|)$$

Complexity of CTL and LTL model checking

LTL model checking problem:

PSPACE-complete and solvable in time

$$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(|\varphi|))$$

LTL with fairness: $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(|\varphi|) \cdot |\mathit{fair}|)$

CTL model checking problem:

solvable in polynomial time (even PTIME-complete)

$$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot |\Phi|)$$

CTL with fairness: $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot |\Phi| \cdot |\mathit{fair}|)$

Recall: LTL fairness assumptions

are conjunctions of **LTL formulas** of the form

- unconditional fairness $\Box\Diamond\phi$
- strong fairness $\Box\Diamond\psi \rightarrow \Box\Diamond\phi$
- weak fairness $\Diamond\Box\psi \rightarrow \Box\Diamond\phi$

where ϕ, ψ are propositional formulas

Reduction of $\models_{fair}$ to $\models$

$\mathcal{T} \models_{fair} \varphi$ iff $\pi \models \varphi$ for all fair paths π in $\mathcal{T}$

iff for all paths π in $\mathcal{T}$:

$\pi \models fair \rightarrow \varphi$

CTL fairness assumptions

conjunctions of "formulas" of the type

- unconditional fairness: $\Box\Diamond\Phi$
- strong fairness: $\Box\Diamond\Psi \rightarrow \Box\Diamond\Phi$
- weak fairness: $\Diamond\Box\Psi \rightarrow \Box\Diamond\Phi$

where Ψ, Φ are CTL state formulas

note: CTL fairness assumptions

- are not CTL (state or path) formulas
- just a syntactic formalism to specify fairness assumptions

Satisfaction relation for CTL with fairness

$s \models_{fair} true$

$s \models_{fair} a$ iff $a \in L(s)$

$s \models_{fair} \neg\Phi$ iff $s \not\models_{fair} \Phi$

$s \models_{fair} \Phi_1 \wedge \Phi_2$ iff $s \models_{fair} \Phi_1$ and $s \models_{fair} \Phi_2$

$s \models_{fair} \exists\varphi$ iff there exists $\pi \in Paths(s)$ with $\boxed{\pi \models fair}$ and $\pi \models_{fair} \varphi$

$s \models_{fair} \forall\varphi$ iff for all $\pi \in Paths(s)$: $\boxed{\pi \models fair}$ implies $\pi \models_{fair} \varphi$

e.g., $s_0\, s_1\, s_2 \ldots \models \Box\Diamond\Phi$ iff $\overset{\infty}{\exists}\, i \geq 0$ s.t. $s_i \models \Phi$

Preprocessing of FairCTL model checking

given: finite transition system $\mathcal{T}$
CTL formula Φ in $\exists$-normal form
CTL fairness assumption $fair$, e.g.,

$$fair = \bigwedge_{1 \le i \le k} \Box\Diamond\Psi_{i,1} \rightarrow \Box\Diamond\Psi_{i,2}$$

question: does $\mathcal{T} \models_{fair} \Phi$ hold ?

preprocessing: apply a standard CTL model checker to evaluate the CTL state subformulas of $fair$

- compute $Sat(\Psi_{i,1})$ and $Sat(\Psi_{i,2})$
- replace $\Psi_{i,1}$ and $\Psi_{i,2}$ with fresh atomic propositions b_i and c_i, respectively

Idea of FairCTL model checking

given: finite transition system $\mathcal{T}$
CTL formula Φ in $\exists$-normal form
CTL fairness assumption ***fair***

question: does $\mathcal{T} \models_{fair} \Phi$ hold **?**

1. ... preprocessing ...
2. Build the parse tree of Φ and process it in bottom-up-manner. Treatment of:
   - ***true***, $a \in AP$, $\wedge$, $\neg$: as for standard **CTL**
   - $\exists\bigcirc$, $\exists\mathsf{U}$: via standard **CTL** model checking
   - $\exists\Box$: via analysis of **SCCs**

Summary: fairness in CTL

CTL fairness assumptions: formulas similar to **LTL**

e.g., $\mathit{fair} = \bigwedge_{1 \le i \le k} (\Box\Diamond\Psi_i \rightarrow \Box\Diamond\Phi_i)$

CTL satisfaction relation with fairness:

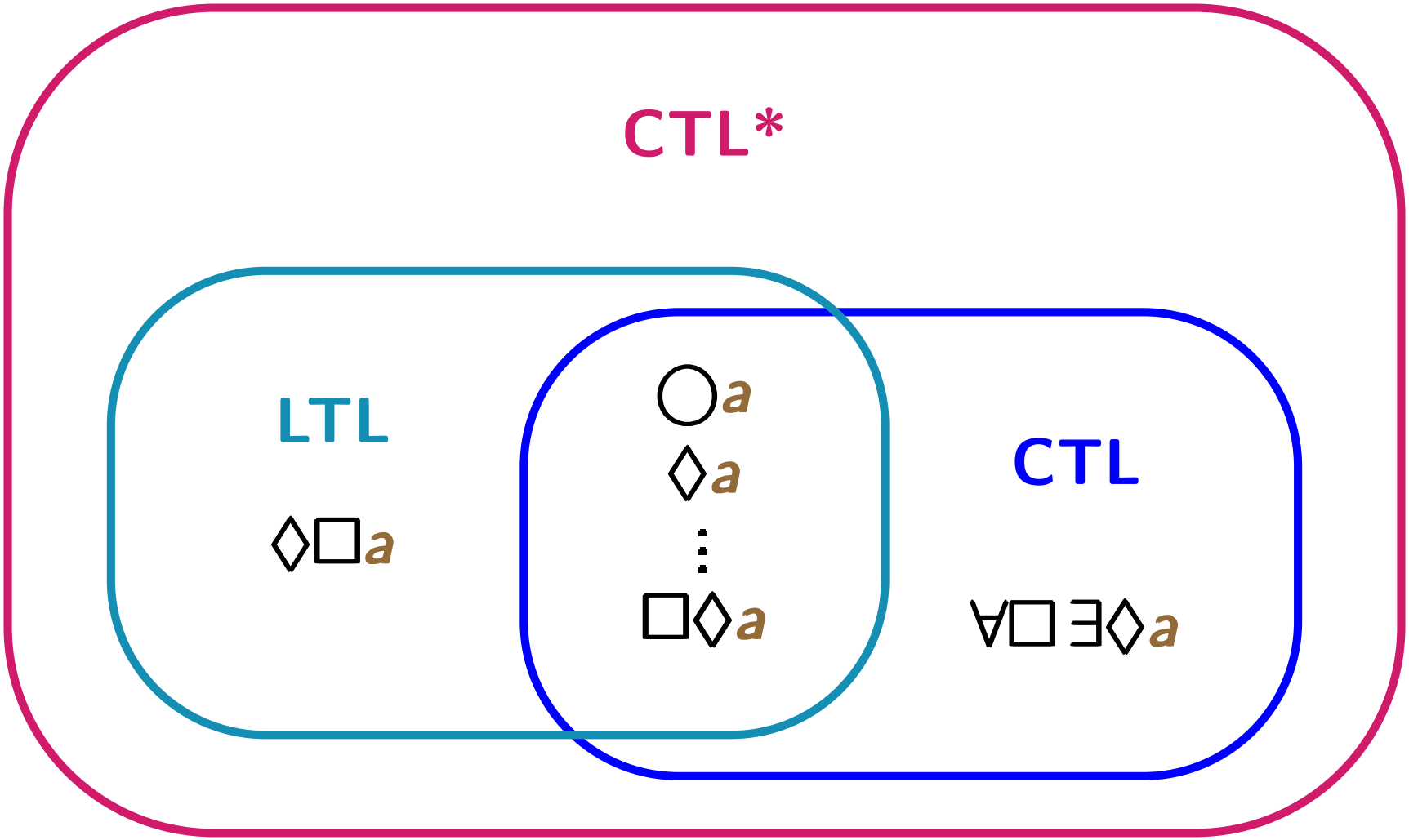
$s \models_{\mathit{fair}} \exists\varphi$ iff there exists $\pi \in \mathit{Paths}(s)$ with $\pi \models \mathit{fair}$ and $\pi \models_{\mathit{fair}} \varphi$

model checking for **CTL** with fairness:

- $\exists\bigcirc$, $\exists\mathbf{U}$, $\forall\bigcirc$, $\forall\Box$ via **CTL** model checker
- analysis of **SCCs** for $\exists\Box$, $\forall\mathbf{U}$
- complexity: $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot |\Phi| \cdot |\mathit{fair}|)$

Overview

Combining LTL and CTL $\rightsquigarrow$ CTL*



Syntax of CTL*

state formulas:

$$\Phi \;::=\; \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi$$

path formulas:

$$\varphi \;::=\; \Phi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \,\mathsf{U}\, \varphi_2$$

derived operators:

- $\vee$, $\rightarrow$, etc.
- eventually, always as in **LTL**:

$$\Diamond\varphi = \mathit{true}\,\mathsf{U}\,\varphi, \quad \Box\varphi = \neg\Diamond\neg\varphi$$

- universal quantification: $\forall\varphi = \neg\exists\neg\varphi$

Semantics of CTL*

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a transition system without terminal states.

define by structural induction:

- a satisfaction relation $\models$ for states $s \in S$ and **CTL*** state formulas
- a satisfaction relation $\models$ for infinite path fragments π in $\mathcal{T}$ and **CTL*** path formulas

Semantics of CTL* state formulas

$s \models true$

$s \models a$ iff $a \in L(s)$

$s \models \neg\Phi$ iff $s \not\models \Phi$

$s \models \Phi_1 \wedge \Phi_2$ iff $s \models \Phi_1$ and $s \models \Phi_2$

$s \models \exists\varphi$ iff there exists a path $\pi \in Paths(s)$ such that $\pi \models \varphi$

satisfaction relation $\models$ for **CTL*** path formulas

Semantics of CTL* path formulas

let $\pi = s_0\, s_1\, s_2 \ldots$ be an infinite path fragment in $\mathcal{T}$

satisfaction relation for **CTL*** state formulas

$\pi \models \Phi$ iff $s_0 \models \Phi$

$\pi \models \neg\varphi$ iff $\pi \not\models \varphi$

$\pi \models \varphi_1 \wedge \varphi_2$ iff $\pi \models \varphi_1$ and $\pi \models \varphi_2$

$\pi \models \bigcirc\varphi$ iff $\mathit{suffix}(\pi, 1) \models \varphi$

$\pi \models \varphi_1 \,\mathbf{U}\, \varphi_2$ iff there exists $j \geq 0$ such that

$\mathit{suffix}(\pi, j) \models \varphi_2$

$\mathit{suffix}(\pi, i) \models \varphi_1$ for $0 \leq i < j$

$\mathit{suffix}(\pi, k) = s_k\, s_{k+1}\, s_{k+2} \ldots$

Examples of CTL*-formulas

mutual exclusion:

safety $\forall\Box(\neg crit_1 \vee \neg crit_2)$

liveness $\forall\Box\Diamond crit_1 \wedge \forall\Box\Diamond crit_2$

progress property, e.g., $\forall\Box(request \rightarrow \Diamond response)$

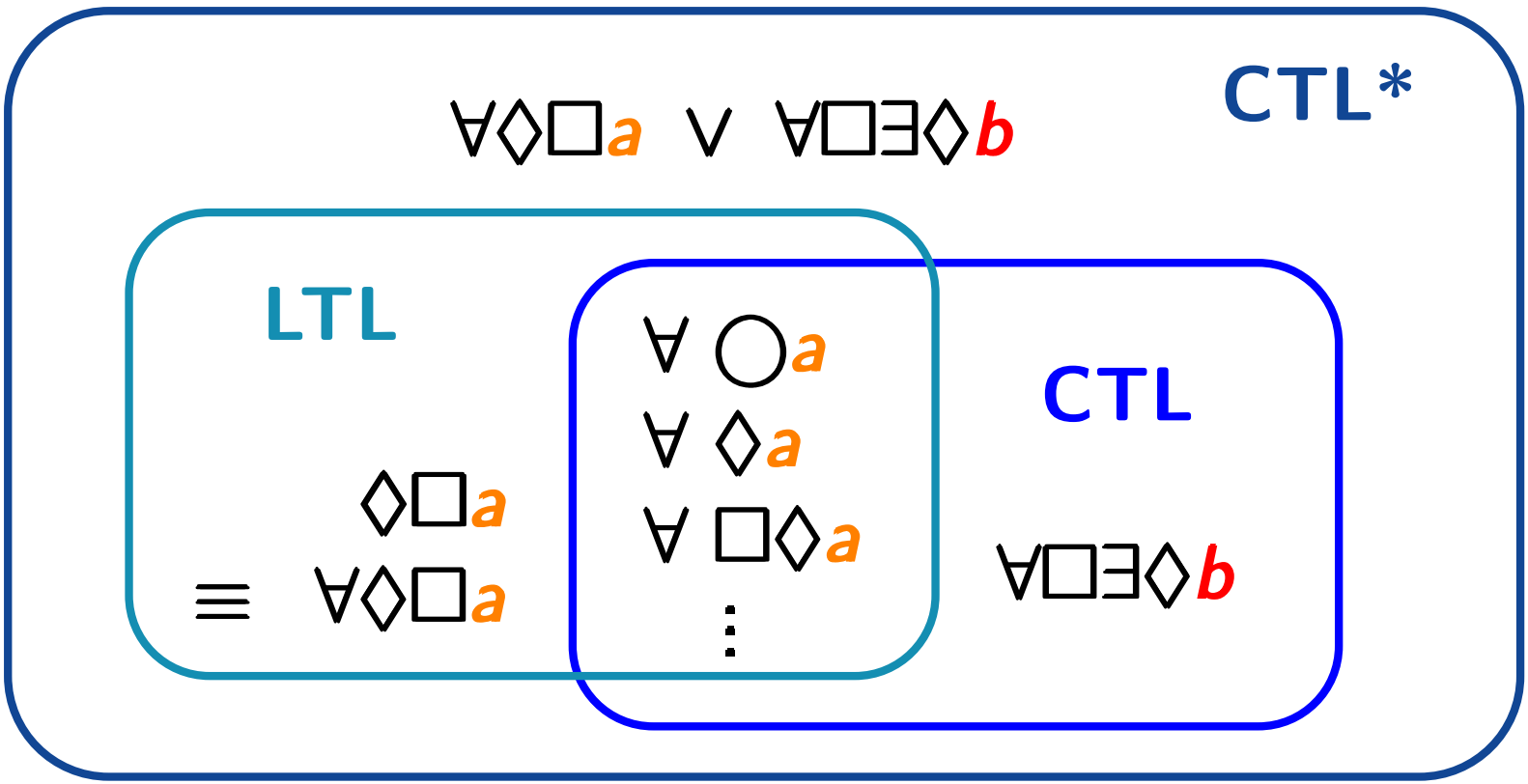
persistence property, e.g., $\forall\Diamond\Box a$

CTL* formulas with existential quantification, e.g.,
Hamilton path problem (for fixed initial state)

$$\exists\Big(\bigwedge_{v\in V}\big(\Diamond v \wedge \Box(v \rightarrow \bigcirc\Box\neg v)\big)\Big)$$

Expressiveness of CTL, LTL and CTL*

- **CTL** is a sublogic of **CTL***
- **LTL** is a sublogic of **CTL***
- **CTL*** is more expressive than **LTL** and **CTL**



Equivalence of CTL*-formulas

$\neg\exists\varphi \equiv \forall\neg\varphi$ e.g., $\neg\exists\Box\Diamond a \equiv \forall\Diamond\Box\neg a$

$\neg\forall\varphi \equiv \exists\neg\varphi$ e.g., $\neg\forall\Box\Diamond a \equiv \exists\Diamond\Box\neg a$

$\forall(\varphi_1 \wedge \varphi_2) \equiv \forall\varphi_1 \wedge \forall\varphi_2$

$\exists(\varphi_1 \vee \varphi_2) \equiv \exists\varphi_1 \vee \exists\varphi_2$

but: $\forall(\varphi_1 \vee \varphi_2) \not\equiv \forall\varphi_1 \vee \forall\varphi_2$

$\exists(\varphi_1 \wedge \varphi_2) \not\equiv \exists\varphi_1 \wedge \exists\varphi_2$

$\forall\Box\Diamond\varphi \equiv \forall\Box\forall\Diamond\varphi$ but: $\forall\Diamond\Box\varphi \not\equiv \forall\Diamond\forall\Box\varphi$

$\exists\Diamond\Box\varphi \equiv \exists\Diamond\exists\Box\varphi$ $\exists\Box\Diamond\varphi \not\equiv \exists\Box\exists\Diamond\varphi$

CTL* model checking

given: finite TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
CTL* formula Φ

question: does $\mathcal{T} \models \Phi$ hold **?**

main procedure as for **CTL**:

```
FOR ALL subformulas Ψ of Φ DO
    compute Sat(Ψ) = {s ∈ S : s ⊨ Ψ}
OD
IF S0 ⊆ Sat(Φ)
    THEN return "yes"
    ELSE return "no"
FI
```

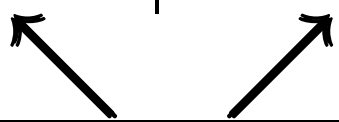
Recursive computation of satisfaction sets

$$
\left.
\begin{aligned}
Sat(true) &= S \\
Sat(a) &= \{s \in S : a \in L(s)\} \\
Sat(\Phi_1 \wedge \Phi_2) &= Sat(\Phi_1) \cap Sat(\Phi_2) \\
Sat(\neg\Phi) &= S \setminus Sat(\Phi)
\end{aligned}
\right\} \text{ as for } \textbf{CTL}
$$

$$
\left.
\begin{aligned}
Sat(\forall\varphi) &= Sat_{LTL}(\varphi) \\
Sat(\exists\varphi) &= S \setminus Sat_{LTL}(\neg\varphi)
\end{aligned}
\right\} \text{ using an } \textbf{LTL} \text{ model checker}
$$

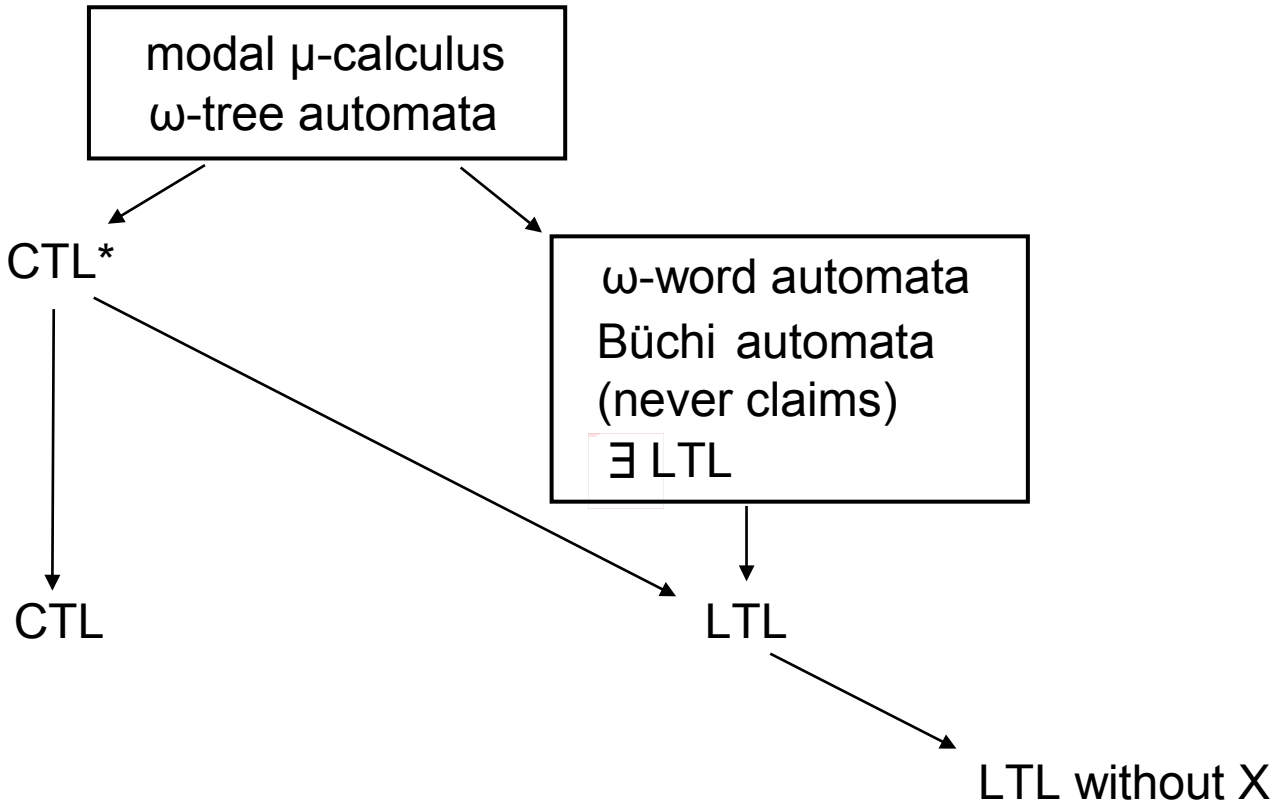
Complexity of CTL/LTL/CTL* model checking

	CTL	LTL and CTL*
	PTIME-complete	*PSPACE*-complete
$\models$	$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \lvert\Phi\rvert)$	$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(\lvert\varphi\rvert))$
$\models_{\mathit{fair}}$	$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \lvert\Phi\rvert \cdot \lvert\mathit{fair}\rvert)$	$\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(\lvert\varphi\rvert) \cdot \lvert\mathit{fair}\rvert)$



model complexity, i.e., for fixed formula:
$\mathcal{O}(\mathit{size}(\mathcal{T}))$

Relationships among logics



same box means ‘equally expressive’
single arrow means ‘more expressive than’
no arrow means ‘expressiveness is not comparable’